EXAM 2 – TAKEHOME SECTION – NOTES, HINTS, COMMENTS

1. Exam Hints and Comments

1.1. **On problem 1.** Part (d) is hard. You might want to let $k = 2$, pick m_1 and m_2, and try to figure out what the sequence converges to. Then try to find something that converges to what you think the limit should be and that is clearly less than $(m_1^n + m_2^n)^{1/n}$. Then find something that is clearly greater than $(m_1^n + m_2^n)^{1/n}$ that also converges to what you think the limit should be.

Parts (a), (b), and (c) were meant to be hints for part (d).

1.2. **On problem 2.** Consider the example $x_n := (-1^n)/n$. Write out the first few terms of the sequence. Write out the definitions of s_1, s_2, and s_3. Then calculate s_1, s_2, and s_3. Do the same with t_1, t_2 and t_3.

Now consider the example $y_n := (-1)^{n+1} + (-1^n)/n$ and do the same thing for this example.

I've suggested this to several people who have asked about this question is office hours. I think that it has helped all of them.

1.3. **Problem 3.** If you're having trouble with part (a), you might want to ask yourself what sequence would converge to $1/e$.

For part (c), it's useful to think first of the sequence

$$(1 + \frac{1}{2n})^{2n}$$

then

$$((1 + \frac{1}{2n})^{2n})^{1/2}$$

and finally

$$(((1 + \frac{1}{2n})^{2n})^{1/2})^3$$

1.4. **Problem 4.** "Prove directly from the definition" means you can't use that Cauchy is the same as convergent.

On the other hand, you will probably need to use Cauchy implies bounded. You don't need to give a proof of this.

Looking at the proof of Theorem 3.2.3 might be helpful.

2. Sequential Criterion for Continuity

Theorem 2.1. *A function $f : A \to \mathbb{R}$ is continuous at $c \in A$ iff for every sequence (x_n) in A that converges to c, the sequence $y_n := f(x_n)$ converges to $f(c)$.*

Proof. First assume that f is continuous at c. Take (x_n) converging to c. Given any $\epsilon > 0$ we must show that there is a K_ϵ such that $n > K_\epsilon$, $|f(x_n) - f(c)| < \epsilon$. By continuity, there is a $\delta > 0$ such that $|x - c| < \delta$ and $x \in A$ implies that $|f(x) - f(c)| < \epsilon$. Since (x_n) converges to c, there must be some H_δ such that

$n > H_\delta$ implies that $|x_n - c| < \delta$. But then we see that if we set K_ϵ to H_δ we obtain $|f(x_n) - f(c)| < \epsilon$ as desired.

Now assume that whenever we have (x_n) converging to c with (x_n) in A. For a contradiction, assume that f is not continuous at c. That is, there is some ϵ_0 such that no matter how small of a δ we pick, there is always some $x \in A$ such that $|x - c| < \delta$ but $abs f(x) - f(c) \geq \epsilon_0$.

Pick x_n so that $|x_n - c| < 1/n$, but $|f(x_n) - f(c)| \geq \epsilon_0$. Then (x_n) converges to c, but $(f(x_n))$ cannot possibly converge to $f(c)$, giving us the contradiction we wanted.

$\square$