EXAM 2 – TAKEHOME SECTION – SOLUTIONS

Problem 1:

(a). Take $m \in \mathbb{N}$. Show directly (that is, from the definitions) that $\left((m^n)^{1/n}\right)$ converges, and find its limit. (Yes, this is easy.)

Proof. Since $\left((m^n)^{1/n}\right)$ is just m, one only has to show that a constant sequence is convergent. Take any $\epsilon > 0$, and let $K_\epsilon = 0$. Then for any $n > K_\epsilon$ one has $|m - m| = 0 < \epsilon$ □

(b). Take $k \in \mathbb{N}$. Show that $\left((k)^{1/n}\right)$ converges and find its limit.

Proof. This is done in the book as Example 3.1.11 (c) □

(c). Take $k, m \in \mathbb{N}$. Show that $\left((km^n)^{1/n}\right)$ converges and find its limit.

Proof. Note that $\left((km^n)^{1/n}\right) = \left((k)^{\frac{1}{n}} (m^n)^{\frac{1}{n}}\right)$. Apply (a), (b), and the theorem about products of convergent sequences, to get that $\lim\left((km^n)^{1/n}\right) = m$ □

(d). Let $m_1, \ldots, m_k$ be elements of $\mathbb{N}$. Show that $\left((m_1^n + \ldots + m_k^n)^{1/n}\right)$ converges and find its limit.

Proof. Say that m_i is the largest of $m_1, \ldots, m_k$. Note that $({m_i}^n)^{1/n} \leq (m_1^n + \ldots + m_k^n)^{1/n} \leq (km_l^n)^{1/n}$. Apply the Squeeze Theorem (and (a) and (c)) to get that $\lim\left((m_1^n + \ldots + m_k^n)^{1/n}\right) = m_i$ □

Problem 2:

(a). Let (x_n) be a bounded sequence, and for each $n \in \mathbb{N}$ let $s_n :=\sup\{x_k : k \geq n\}$, and let $t_n :=\inf\{x_k : k \geq n\}$. Prove that (s_n) and (t_n) are monotone and convergent. We call $\lim(s_n)$ the limit superior of (x_n), or $\text{limsup}(x_n)$, and we call $\lim(t_n)$ the limit inferior of (x_n) or $\text{liminf}(x_n)$.

Proof. First we will show that s_n is decreasing, and then that it is bounded. By definition, $s_n = \sup\{x_k : k \geq n\}$ and $s_{n+1} = \sup\{x_k : k \geq n+1\}$. Since $\{x_k : k \geq n+1\} \subseteq \{x_k : k \geq n\}$ it follows that $s_{n+1} \leq s_n$. Since (x_n) is bounded, s_n is bounded above by $\sup\{x_n : n \in \mathbb{N}\}$. It is bounded below by $\inf\{x_n : n \in \mathbb{N}\}$. To see this last fact, note that for each k, $\inf\{x_n : n \in \mathbb{N}\} \leq x_k \leq \sup\{x_i : k \geq i\} = s_k$.

The argument that t_n is increasing and bounded works just the same. □

(b). Prove that if limsup(x_n) =liminf(x_n), then (x_n) is convergent.

Proof. Note that $t_n = \inf\{x_k :\geq n\} \leq x_n \leq \sup\{x_k : k \geq n\} = s_n$. Now apply the Squeeze Theorem. □

(c). Give an example of a convergent sequence (x_n) such that for all m and k, $s_m \neq x_k$. (Yes, this is easy.)

Proof. Any increasing, bounded sequence will do. For instance, take $x_n = -1/n$. Then $s_n = 0$ for all n, but 0 never appears in the sequence (x_n).

It is important to keep this example in mind to avoid making the mistake of assuming that (s_n) is a subsequence of (x_n), which is particularly tempting in the next part. □

(d). Show that if (x_n) is convergent then $\limsup(x_n) = lim(x_n)$.

Proof. Suppose that $K_{\epsilon/2}$ is such that for all $n > K_{\epsilon/2}$, $|x_n - x| < \epsilon/2$. Then the supremum of all x_n with $n > K_{\epsilon/2}$ is at most $x + \epsilon/2$. Thus $|x - s_n| < \epsilon$, so (s_n) converges to x, and thus $\limsup(x_n) = x$. □

(e). Say (a_n) and (b_n) are bounded sequences. Show that $\limsup(a_n + b_n)$ exists and that $\limsup(a_n + b_b) \leq \limsup(a_n) + \limsup(b_n)$.

Proof. Let $S_m := \sup\{a_k + b_k : k > m\}$, let $s_m := \sup\{a_k : k > m\}$, and let $\sigma_m := \sup\{b_k : k > m\}$. Suppose that $S_m > s_m + \sigma_m$. Then there is some $l > m$ such that $a_l + b_l > s_m + \sigma_m$. But $s_m > a_l$ and $\sigma_m > b_l$, so this is a contradiction. Thus $S_m \leq s_m + \sigma_m$ and thus $lim(S_m) \leq lim(s_m + \sigma_m)$. □

(f). Find an example where the inequality above is strict.

Let $(a_n) := (-1, 1, -1, 1, -1, \ldots)$ and let $(b_n) := (1, -1, 1, -1, \ldots)$.

Problem 3. Find the limits of the following sequences: (Hint: the answers are $1/e$, e, and $e^{3/2}$.)

(a). (x_n), where $x_n := \left(\frac{n}{n+1}\right)^n$

Here, the important thing to notice is that $x_n = 1/e_n$, where $e_n := (1 + 1/n)^n$. Since (e_n) converges to e, (x_n) converges to $1/e$.

(b). (y_n), where $y_n := (1 + 1/n)^{n+1}$

Here, we right y_n as $(1 + 1/n)^n \cdot (1 + 1/n) = e_n \cdot (1 + 1/n)$. Thus $\lim(y_n) = \lim(e_n) \cdot \lim(1 + 1/n) = e \cdot 1$.

(c). (z_n), where $z_n := (1 + 1/2n)^{3n+1}$

First note that the subsequence e_{2n} of e_n is just $(1 + 1/2n)^{2n}$, which thus also converges to e. It is a theorem in the book that if (a_n) converges to a, and if one denotes the positive square root of a_n as $\sqrt{a_n}$, then $\sqrt{a_n}$ converges to $\sqrt{a}$. Thus we conclude that $\left((1 + 1/2n)^{2n}\right)^{\frac{1}{2}} = (1 + 1/2n)^n$ converges to $\sqrt{e}$. Finally, $(1 + 1/2n)^n \cdot (1 + 1/2n)^n \cdot (1 + 1/2n)^n = (1 + 1/2n)^{3n}$ converges to $\sqrt{e}\sqrt{e}\sqrt{e} = e^{3/2}$.

Now we just apply the same reasoning as in part (b): $z_n = (1 + 1/2n)^{3n+1} = (1 + 1/2n)^{3n} \cdot (1 + 1/2n)^1$. Thus

$$\lim(z_n) = \lim\left((1 + 1/2n)^{3n}\right) \cdot \lim\left(1 + 1/2n\right) = e^{3/2} \cdot 1$$

Problem 4: Show directly from the definition that if (x_n) and (y_n) are Cauchy sequences than so are:

(a). $(x_n + y_n)$

Proof. Since (x_n) is Cauchy, we have H_1 such that for $n, m > H_1$, $|x_n - x_m| < \epsilon/2$. Likewise, we have some H_2 such that for $n, m > H_2$, $|y_n - y_m| < \epsilon/2$. Let H be whichever of H_1 and H_2 is greater. By the triangle inequality, we have that $|(x_n + y_n) - (x_m + y_m)| \le |x_n - x_m| + |y_n - y_m|$. If $n, m > H$, we have that $|x_n - x_m| + |y_n - y_m| < \epsilon/2 + \epsilon/2$. Putting these two facts together, we see that for $n, m > H$, $|(x_n + y_n) - (x_m + y_m)| < \epsilon$, which is what we needed to show in order to proof that $(x_n + y_m)$ is a Cauchy sequence. □

(b). $(x_n \cdot y_n)$

Proof. First we note that Cauchy sequences are bounded, so we have M_1 such that $|x_n| < M_1$ for all n, and M_2 such that $|y_n| < M_2$ for all n. Choose H_1 such that for all $n, m > H_1$, $|x_n - x_m| < \epsilon/M_2$ and choose H_2 such that for all $n, m > H_2$, $|y_n - y_m| < \epsilon/M_2$. Now we have $|x_n y - n - x_m y - m| = |x_n y_n - x_m y_n + x_m y_n - x_m y_m| = |x_n - x_m| y_n + |y_n - y_m| x_m \le |x_m - x_m| M_2 + |y_n - y_m| M_1 < \frac{\epsilon}{M_2} M_2 + \frac{\epsilon}{M_1} M_1 = \epsilon$ □

Problem 5:

(a). Let $f : \mathbb{R} \to \mathbb{R}$, let I be an open interval, and let $c \in I$. Let f_1 be the restriction of f to I. Show that f_1 has a limit at c iff f has a limit at c.

Proof. First we show that if f is continuous, then so is f_1. Take $c \in I$. Take $\epsilon > 0$. Since f is a continuous, there is a $\delta > 0$ such that $|x - c| < \delta$ implies that $|f(x) - L| < \epsilon$. And thus it is also true that for x such that $x \in I$ and $|x - c| < \delta$, $|f(x) - L| < \epsilon$. But $|f(x) - L| = |f_1(x) - L|$ for $x \in I$, so we also have $|f_1(x) - L| < \epsilon$. Note that in this part of the proof, I can be any set, not just an open interval.

Now assume that f_1 is continuous. Take $c \in I$, take $\epsilon > 0$. Since f_1 is continuous, there is a $\delta > 0$ such that $x \in I$ and $|x - c| < \delta$ implies that $|f_1(x) - L| < \epsilon$. But what about the x such that $|x - c| < \delta$ but x is not in I? We know nothing about these x. This is where we need to use the fact that I is open. By the definition of open, there is some $\beta > 0$ such that $V_\beta(c)$ is entirely contained in I. Take γ equal to the minimum of β and δ. Now, for all x such that $|x - c| < \gamma$, we have $|f_1(x) - L| < \epsilon$ and for all of these x, $f_1(x) = f(x)$. Thus we conclude that $|f(x) - L| < \epsilon$ and, thus, f is continuous at c. □

(b). Let $f : \mathbb{R} \to \mathbb{R}$, let I be an closed interval, and let $c \in I$. Let f_2 be the restriction of f to I. Show that if f has a limit at c then so does f_2.

Proof. Note that this uses only the part of the proof above that did not depend on I being open. □

(c). Let f_2, c, and I be as above. Show that f_2 may have a limit at c even though f does not. (And go back to make sure that your proof in part (a) doesn't work for I closed.)

Proof. Let $f(x)$ be $\operatorname{sgn}(x)$. That is, let $f(x) = 1$ for positive x, let $f(0) = 0$, and let $f(x) = -1$ for negative x. Let $I = [0, 1]$ and let f_2 be the restriction of f to I. It is easy to see that for any c in $[0, 1]$, the limit of f_2 as x approaches c is 1, but there is no limit of f as x approaches 0. □

Problem 6:

(a). Formulate an equivalent definition of continuity using ϵ and δ neighborhoods rather than absolute values (i.e. prove Theorem 5.1.2).

Proof. We will show that $f : A \to \mathbb{R}$ is continuous at $c \in A$ if and only if for each $\epsilon > 0$ there is a $\delta > 0$ such that $f(V_\delta(c) \cap A) \subseteq V_\epsilon(f(c))$. By definition, f is continuous at c iff for any $\epsilon > 0$, there is a $\delta > 0$, such that $|x - c| < \delta$ and $x \in A$ implies that $|f(x) - f(c)| < \epsilon$. Noting that that the definition of an ϵ-neighborhood of a is that $x \in V_\epsilon(a)$ iff $|x - a| < \epsilon$, we can rewrite the previous sentence as follows: f is continuous at c iff for any $\epsilon > 0$, there is a $\delta > 0$, such that $x \in V_\delta(c) \cap A$ implies that $f(x \in V_\epsilon(f(c))$. And since the definition of "$f(C) \subseteq D$" is "if x is in C then $f(x)$ is in D", we can rewrite the previous sentence once again, this time as: $f : A \to \mathbb{R}$ is continuous at $c \in A$ if and only if for each $\epsilon > 0$ there is a $\delta > 0$ such that $f(V_\delta(c) \cap A) \subseteq V_\epsilon(f(c))$, which is what we wanted. □

(b). If a function, f, from $\mathbb{R}$ to $\mathbb{R}$ is continuous, then for any open set $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is an open set.

Proof. Take $x \in f^{-1}(U)$. Then $f(x) \in U$. By the definition of open, there is some ϵ such that $V_\epsilon(f(x)) \subseteq U$. By the definition of continuity from part (a), there is δ such that $f(V_\delta(x)) \subseteq V_\epsilon(f(x))$. Thus $V_\delta(x)$ is contained in $f^{-1}(U)$. Since x was arbitrary, we see that around any point of $f^{-1}(U)$ there is a δ-neighbor contained in $f^{-1}(U)$, which is what it means to be open. □

(c). If a function, f, from $\mathbb{R}$ to $\mathbb{R}$ has the property that for any open set $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is an open set, then f is continuous.

Proof. Take $V_\epsilon(f(x))$. By the definition of continuity from (b), we need to find some $V_\delta(x)$ such that $f(V_\delta(x)) \subseteq V_\epsilon(f(x))$. We note that, as an open interval, $V_\epsilon(f(x))$ is an open set. Thus its inverse image under f is open. Thus its inverse image contains some $V_\delta(x)$. □

(d). Give a example of a continuous function, and an open interval I, such that $f(I)$ is not itself an open set.

Take any constant function.

(e). Prove that a function is continuous iff the inverse image of every closed set is closed.

Proof. All one really needs to do is to note that for any set A, $f^{-1}(A^C)$ is the same set as $f^{-1}(A)^C$. Then one has f is continuous iff $f^{-1}(U)$ is open for any open set U iff $f^{-1}(U)^C$ is closed for any open set U iff $f^{-1}(U^C)$ is closed for any open set U iff $f^{-1}(K)$ is closed for any closed set K. □

Problem 7: Say a point $c \in A$ is an *isolated* point of A iff there is some $V_\delta(c)$ such that c is the only point of A contained in $V_\delta(c)$. Show that c is an isolated point of A if and only if it is not a cluster point of A

Proof. Recall that a point c is a cluster point of A iff for every $\epsilon > 0$, $V_\epsilon(c)$ contains some $a \in A$ such that $a \neq c$. Thus a point $c \in A$ is not a cluster point iff for some ϵ the only point in $V_\epsilon(c)$ from A. But this is just what it means for c to be an isolated point. $\square$