EXAM 3 – TAKEHOME SECTION

(due Friday, December 2nd)

Problem 1: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) := x^2 \sin(1/x)$ for $x \neq 0$ and $f(x) := 0$ for $x = 0$. Although we have not proven it in class, you may use without proof the fact that the derivative of $\sin(x)$ is $\cos(x)$.

(a). Calculate $f'(c)$ for $c \neq 0$.

(b). Prove that for any $g : \mathbb{R} \to \mathbb{R}$ such that $|g(x)| \leq x^2$, g is differentiable at zero.

(c). Prove that a function that is differentiable everywhere in $\mathbb{R}$ need not have a continuous derivative.

Problem 2: Let $f := \mathbb{R} \to \mathbb{R}$.

(a). Show that if f is strictly increasing or strictly decreasing, then f is injective.

(b). Show that if f is continuous and injective then f is either strictly increasing or strictly decreasing.

Problem 3. Recall that a function is continuous iff inverse images of open set areopen. In this problem we will consider when images of open sets are open.

(a). Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ where $f(I)$ is an open interval whenever I is an a open interval.

(b). Note that the example you gave in (b) is either strictly increasing or strictly decreasing. Prove that this is always the case. (Hint: Use Problem 2)

Problem 4: Define *upper semi – continuous* as follows: A function $f : A \to \mathbb{R}$ is said to be upper semicontinuous at $a \in A$ iff for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $x \in V_\delta(a)$, $f(x) < f(a) + \epsilon$. A function is said to be upper semi-continuous iff it is upper semi-continuous at every point where it is defined.

(a). Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) := x$ for $x < 0$ and $g(x) := x + 1$ for $x \geq 0$. Let $h : \mathbb{R} \to \mathbb{R}$ be given by $h(x) := x$ for $x \leq 0$ and $h(x) := x + 1$ for $x > 0$. Show that g is upper semi-continuous but h is not.

(b). Prove that if (x_n) converges to x, and f is upper semi-continuous, and defined on both the sequence x_n, and the point x, then $\lim(f(x_n)) \leq f(x)$.

(c). Prove that if $f : [a, b] \to \mathbb{R}$ is upper semi-continuous, then f is bounded above and attains its maximum at some $c \in [a, b]$. (This proof will be long, but if you understand section 5.3, it won't be hard.)

Problem 5: Prove that $|\sin x - \sin y| \leq |x - y|$. (Hint: use the Mean Value Theorem).

Problem 6: Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Prove that if $lim_{x \to a} f'(x)$ exists and equals c, then $f'(a)$ exists and equals c.