Math 231 Section B
Summer 2009
Midterm Exam II
July 22, 2009

- You have 50 minutes to complete this exam.
- Show ALL works.
- Please REMOVE hat and sunglasses. Turn off ALL electronic devices.
- Good Luck!

1	2	3	4	5	EC	Total

Name __Solutions__________

UID ___000-00-000

Signature__Solutions________

1. (15 points) Find the limit, if any, of the following sequence:

$$a_n = \frac{cos(\sqrt{n})}{n}$$

$$-1 \leq \cos\sqrt{n} \leq 1$$

$$\therefore \frac{-1}{n} \leq \frac{\cos\sqrt{n}}{n} \leq \frac{1}{n}$$

and since $\frac{1}{n} \to 0$, $\frac{-1}{n} \to 0$

By squeeze theorem,

$$\frac{\cos\sqrt{n}}{n} \to 0 \quad \text{as} \quad n \to \infty \quad //$$

2.(15points) Determine convergence behavior of the following series. (Be sure to show ALL works)

$$\sum_{k=1}^{\infty} \frac{k^3 - 2k + 2}{4k^3 + 15k - 18}$$

$$\lim_{k \to \infty} \frac{k^3 - 2k + 2}{4k^3 + 15k - 18}$$

$$= \lim_{k \to \infty} \frac{1 - \frac{2}{k^2} + \frac{2}{k^3}}{4 + \frac{15}{k^2} - \frac{18}{k^3}}$$

$$= \frac{1}{4} \neq 0$$

By k-TH term test, the series

diverges //

3. Consider the function $ln(x+2)$

(a) (15 points) Find its power series representation.
(b) (15 points) Determine its interval and radius of convergence.

ⓐ

$$\ln(x+2) = \int \frac{1}{x+2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{1+\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int \sum_{k=0}^{\infty}\left(-\frac{x}{2}\right)^k dx$$

$$= \frac{1}{2}\int \sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k x^k\,dx$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{2}\right)^k\int x^k\,dx$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\right)^k}{k+1}\,x^{k+1} \quad //$$

alternative:

$$\ln x = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(x-1)^k$$

$$\therefore \ln(x+2) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(x+1)^k$$

converges ~~on~~ when

$x+2 \in (0,2]$

or $x \in (-2,0]$

ⓑ

$$L(x) = \lim_{k\to\infty}\frac{\frac{\left(\frac{1}{2}\right)^{k+1}}{k+2}|x|^{k+2}}{\frac{\left(\frac{1}{2}\right)^k}{k+1}|x|^{k+1}}$$

$$= \frac{1}{2}|x|$$

$$\frac{1}{2}|x| < 1 \iff |x| < 2 \iff -2 < x < 2$$

$x = 2$

$$\sum_{k=0}^{\infty} \frac{(-1)^k \frac{1}{2^k}}{k+1} \cdot 2^{k+1}.$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2}{k+1} = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$$

$\frac{1}{k+1} \to 0$, and $\frac{1}{k+1} - \frac{1}{k+2}$

$$= \frac{-1}{(k+1)(k+2)} < 0$$

∴ decreasing.

A.S.T ⇒ series converges.

$x = -2$.

$$\sum_{k=0}^{\infty} \frac{(-1)^k \frac{1}{2^k}}{k+1} (-2)^{k+1} = \sum_{k=0}^{\infty} \frac{\overset{1}{\cancel{(-1)^k (-1)^k}}}{\cancel{2^k}(k+1)} 2^{\cancel{k}+1}$$

$$= \sum_{k=0}^{\infty} \frac{2}{k+1} = \infty$$

(harmonic series)

∴ interval of convergence $= (-2, 2]$.

radius of conv. $= 2$

4. (15 points) Given the following series

$$\sum_{k=1}^{\infty}(-1)^k\frac{k+1}{k^2}$$

Assume this series converges. Find the LEAST number of terms required to add so that the partial sum has an error less than $\frac{1}{5}$.

$$R_n = |S - S_n| \leq a_{n+1} = \frac{n+2}{(n+1)^2} \leq \frac{1}{5}$$

$n=1$: $\frac{3}{4} \not\leq \frac{1}{5}$

$n=2$: $\frac{4}{9} \not\leq \frac{1}{5}$

$n=3$: $\frac{5}{16} \not\leq \frac{1}{5}$

$n=4$: $\frac{6}{25} \not\leq \frac{1}{5}$

$n=5$: $\frac{7}{36} \leq \frac{1}{5}$ ✓

$\therefore$ need at least 5 terms

(anything ≥ 5 (provided everything is correct) : 8 points)

5. Taylor Series...

(a) (10 points) Find Taylor series for e^{-x^2} about $x = 0$

(b) (10 points) Assuming the series you found in (a) is actually e^{-x^2}, compute the following limit

$$\lim_{x \to 0} \frac{e^{-x^2} - 1 + x^2}{4x^4}$$

ⓐ $$e^{x} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\therefore \quad e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}$$

ⓑ $$\frac{e^{-x^2} - 1 + x^2}{4x^2}$$

$$= \lim_{x \to 0} \frac{(\cancel{1} - \cancel{x^2} + \frac{x^4}{2} - \cdots)}{4x^4}$$

$$= \lim_{x \to 0} \left[\frac{1}{8} - x (\cdots) \right]$$

$$= \frac{1}{8}$$

(Extra Credit - 10 points) (Show ALL works) Given a sequence a_n such that $\lim_{n\to\infty} =$

3. Determine convergence behaviors for the following two series:
   (a) $\sum_{k=2}^{\infty} \frac{a_k}{k-1}$
   (b) $\sum_{k=1}^{\infty} \frac{(k-6.\ )a_k}{k^3+k+3}$

$$a_k \to 3.$$

$\therefore$ $\exists$ K Large enough s.t.

$$2.9 \leq a_k \leq 3.1$$

ⓐ $$\sum_{k=K}^{\infty} \frac{a_k}{k-1} \geq \sum_{k=K}^{\infty} \frac{2.9}{k-1} \underset{\uparrow}{\boxed{\geq}} \sum_{k=K}^{\infty} \frac{2.9}{k} = \infty$$

$$\frac{2.9}{k-1} \geq \frac{2.9}{k}$$

$\therefore$ $\sum_{k=K}^{\infty} \frac{a_k}{k-1}$ diverges $\Rightarrow$ $\sum_{k=2}^{\infty} \frac{a_k}{k-1}$ diverges.

ⓑ $$\sum_{k=K}^{\infty} \frac{(k-6.\ )a_k}{k^3+k+3} \qquad 3.1 \sum_{k=K}^{\infty} \frac{(k-6)}{k^3+k+3}$$

compare to $\frac{k}{k^3} = \frac{1}{k^2}$

$$\lim_{k\to\infty} \frac{\frac{k-6}{k^3+k+3}}{1/k^2} = 1 > 0,$$ $\therefore$ L.C.T. applies

and since $\sum_{k=K}^{\infty} \frac{1}{k^2}$ converges ($P = 2 > 1$)

$\therefore$ $3.1 \sum_{k=K}^{\infty} \frac{k-6}{k^3+k+3}$ converges $\Rightarrow$ $\sum_{k=2}^{\infty} \frac{(k-6)a_k}{k^3+k+3}$ converges