HW 10.11 (Graded) Solutions

§8.7: 23, 39, 56

(23) $\ln x \text{ "="} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$, $1 \le x \le 2$.

$$\exists \quad R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-1)^{n+1}$$

$$= \frac{(-1)^{n+1} n!\, z^{-(n+1)}}{(n+1)!}(x-1)^{n+1}$$ for some z s.t. $1 \le z \le x$.

$$= \frac{(-1)^{n+1}}{(n+1) z^{n+1}}(x-1)^{n+1}$$

$$|R_n(x)| = \frac{1}{(n+1)|z|^{n+1}} |x-1|^{n+1}$$

$\left\{ \begin{array}{l} 1 \le x \le 2 \Rightarrow |x-1| \le 1 \\ 1 \le z \Rightarrow 1 \le |z| \end{array} \right.$ $\le \frac{1}{n+1} \xrightarrow{n\to\infty} 0$

$\therefore R_n(x) \to 0$ as $n \to \infty$. //

(39) want to show that $f'(0) = f''(0) = 0$.
can't differentiate e^{-1/x^2} and plug in $x=0$,
since e^{-1/x^2} is defined away from 0.
Instead, we compute them from raw definitions:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - \underbrace{f(0)}_{=0}}{h}$$

$$= \lim_{h\to 0} \frac{e^{-1/h^2}}{h} = 0$$ since $h \ne 0$ ⟨given fact⟩

$$f''(0) = \lim_{h\to 0} \frac{f'(h) - \underbrace{f'(0)}_{=0}}{h}$$ $=0$, just computed

$$= \lim_{h\to 0} \frac{\frac{d}{dx} e^{-1/x^2}\big|_{x=h}}{h} = \lim_{h\to 0} \frac{\frac{2}{h^3} e^{-1/h^2}}{h} = 2 \lim_{h\to 0} \frac{e^{-1/h^2}}{h^3} = 0$$ //

(56) note that if n is an integer, then $\binom{n}{k} = 0$, for all $k > n$, since

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \hookleftarrow k \text{ terms on top.}$$

if $k > n$ & n is an integer, 0 must be among the k terms on top. < start from integer n & goes down more than k steps ... must get 0 somewhere >

$\therefore$ Here,

$n = 2$. $\binom{n}{k} = 0 \quad \forall k > 2$.

$$\therefore (1+x)^2 = 1 + \sum_{k=1}^{2} \binom{2}{k} x^k$$

$$= 1 + \binom{2}{1} x + \binom{2}{2} x^2 = 1 + 2x + x^2$$

$$(1+x)^3 = 1 + \binom{3}{1} x + \binom{3}{2} x^3 + \binom{3}{3} x^3 = 1 + 3x + 3x^2 + x^3$$

$$(1+x)^4 = 1 + \binom{4}{1} x + \binom{4}{2} x^2 + \binom{4}{3} x^3 + \binom{4}{4} x^4$$

$$= 1 + 4x + 6x^2 + 4x^3 + x^4 \;/\!/$$

§8.8: 18, 23, 33, 36, 41

(18) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, $e^{\sqrt{x}} = e^{x^{\frac{1}{2}}} = \sum_{k=0}^{\infty} \frac{x^{\frac{k}{2}}}{k!}$

$$\int_0^1 e^{\sqrt{x}}\,dx = \int_0^1 \sum_{k=0}^{\infty} \frac{x^{\frac{k}{2}}}{k!}$$

$$= \sum_{k=0}^{\infty} \int_0^1 \frac{x^{\frac{k}{2}}}{k!}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{x^{\frac{k}{2}+1}}{k!\left(\frac{k}{2}+1\right)} \Bigg|_0^1$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!\left(\frac{k}{2}+1\right)}$$

$$\approx \frac{1}{1\cdot 1} + \frac{1}{2(1+1)} + \frac{1}{3!\left(\frac{3}{2}+1\right)} + \frac{1}{4!\left(\frac{4}{2}+1\right)} + \frac{1}{5!\left(\frac{5}{2}+1\right)} + \cdots$$

(23) $m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}} = m_0\left(1 - \frac{v^2}{c^2}\right)^{\frac{-1}{2}}$

$$= m_0 \sum_{k=0}^{\infty} \binom{\frac{-1}{2}}{k} \left(-\frac{v^2}{c^2}\right)^k$$

(note $\left|\frac{v^2}{c^2}\right| < 1$) $\therefore$ ok to use binomial series.

$$= m_0\left(1 - \binom{\frac{-1}{2}}{1}\left(\frac{v^2}{c^2}\right) + \cdots\right)$$

$$= m_0\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right)$$

$$= m_0 + \left(\frac{m_0}{2c^2}\right)v^2 + \cdots$$

$$\therefore \quad m \approx m_0 + \left(\frac{m_0}{2c^2}\right)v^2.$$

$$1.1\,m_0 = m_0 + \left(\frac{m_0}{2c^2}\right)v^2 \Rightarrow 0.1\,\cancel{m_0} = \frac{\cancel{m_0}}{2c^2}v^2$$

OR $v = \sqrt{0.2}\,c$ //

(33) $f(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{\frac{-1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{-1}{2}}{k} (-x)^k$

$\binom{\frac{-1}{2}}{0} = 1, \quad \binom{\frac{-1}{2}}{1} = \frac{-1}{2}, \quad \binom{\frac{-1}{2}}{2} = \frac{\frac{-1}{2}\left(\frac{-1}{2}\right)}{2} = \frac{3}{8}$

$\binom{\frac{-1}{2}}{3} = \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!} = -\frac{15}{16}$

$\binom{\frac{-1}{2}}{4} = \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)\left(\frac{-1}{2}-3\right)}{4!} = \frac{35}{128}$

$\therefore$ First Five Terms $= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{15}{16}x^3 + \frac{35}{128}x^4$

(36) $f(x) = (1+x^2)^{\frac{4}{5}} = \sum_{k=0}^{\infty} \binom{\frac{4}{5}}{k} (x^2)^k = \sum_{k=0}^{\infty} \binom{\frac{4}{5}}{k} x^{2k}$

$\binom{\frac{4}{5}}{0} = 1, \quad \binom{\frac{4}{5}}{1} = \frac{4}{5}, \quad \binom{\frac{4}{5}}{2} = \frac{\left(\frac{4}{5}\right)\left(\frac{4}{5}-1\right)}{2} = -\frac{1}{25}$

$\binom{\frac{4}{5}}{3} = \frac{\left(\frac{4}{5}\right)\left(\frac{4}{5}-1\right)\left(\frac{4}{5}-2\right)}{6} = \frac{4}{125}, \quad \binom{\frac{4}{5}}{4} = \frac{\left(\frac{4}{5}\right)\left(\frac{4}{5}-1\right)\left(\frac{4}{5}-2\right)\left(\frac{4}{5}-3\right)}{24}$

$= -\frac{11}{625}$

First 5 Terms $= 1 + \frac{4}{5}x^2 - \frac{1}{25}x^4 + \frac{4}{125}x^3 - \frac{11}{625}x^4 \cdots$

(41) $\sin^{-1}x = \int \frac{1}{\sqrt{1-x^2}}\,dx = \int (1-x^2)^{\frac{-1}{2}}\,dx$

$= \int \sum_{k=0}^{\infty} \binom{\frac{-1}{2}}{k} (-x^2)^k\,dx$

$= \sum_{k=0}^{\infty} \binom{\frac{-1}{2}}{k} (-1)^k \int x^{2k}\,dx$

$= \sum_{k=0}^{\infty} \binom{\frac{-1}{2}}{k} (-1)^k \frac{x^{2k+1}}{2k+1}$

$= \sum_{k=0}^{\infty} \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)\cdots\left(\frac{-1}{2}-k+1\right)}{k!} \frac{(-1)^k}{2k+1} x^{2k+1}$

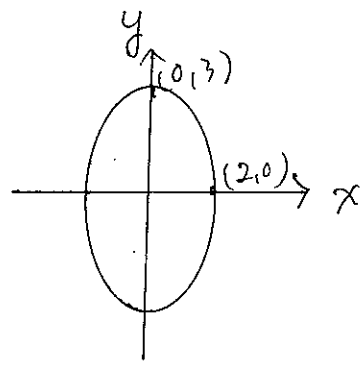
§9-1 : 1, 8

(1) $\begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases} \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

no restriction on output of sin & cos

$\mathbin{/\!/}$

$\frac{x^2}{2^2} + \frac{y^2}{3^2} \rightsquigarrow$ ellipse



(8) $\begin{cases} x = t^2 - 1 \\ y = t^2 + 1 \end{cases}$ $\left\langle \text{note: } t^2 \geq 0. \quad \therefore\ x \geq -1,\ y \geq 1 \right\rangle$

$\Downarrow$

$x + 1 = y - 1$ OR $y = x + 2$ w/ $x \geq -1$, $y \geq 1$