HW12 (Graded) Solutions

39.1 : 26, 30, 32, 36, 44

(26) $\begin{cases} x = t-1 \\ y = t^3 \end{cases} \Rightarrow y = (x+1)^3$ $\therefore$ obviously, D

(30) $\begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases} \Rightarrow \dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1$ $\therefore$ ellipse $\Rightarrow$ F.

(32) Many possible parametrizations ...

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For example, want the total time spent on the segment to be 1, ie $0 \le t \le 1$ & $t=0$ @ $(3,1)$ & $t=1$ @ $(1,3)$, take

$$\begin{cases} x = 3 + (1-3)t = 3 - 2t \\ \quad \langle\, x(0) = 3 \quad \& \quad x(1) = 2 \,\rangle \\ y = 1 + (3-1)t = 1 + 2t \\ \quad \langle\, y(0) = 1 \quad \& \quad y(1) = 3 \,\rangle. \end{cases}$$

(36) $y = 2x^2 - 1$ from $(0,-1)$ to $(2,7)$

let $x = t$, $y = 2t^2 - 1$.

& $0 \le t \le 2$

(44) $\begin{cases} x = t^2+3 \\ y = t^3+t \end{cases}$ & $\begin{cases} x = 2+s \\ y = 1-s \end{cases}$

at point of intersection, we have

$$\begin{cases} t^2+3 = 2+s \\ t^3+t = 1-s \end{cases} \quad \text{OR} \quad \underbrace{t^2+3-2}_{= t^2+1} = \underbrace{-(t^3+t-1)}_{= -t^3-t+1}$$

$$\Rightarrow \quad t^3+t^2+t=0$$

$$t(t^2+t+1)=0$$

$$t=0$$

$$\Rightarrow \quad s=1$$

$\therefore$ intersect at $(3,0)$ //

§9-2: 2, 6, 10, 14, 18, 22, 26, 30

(2) $\begin{cases} x = t^3 - t \\ y = t^4 - 5t^2 + 4 \end{cases} \Rightarrow \begin{cases} x' = 3t^2 - 1 \\ y' = 4t^3 - 10t \end{cases}$

ⓐ at $t = -1$, $\quad x'(-1) = 2$, $y'(-1) = 6$. $\quad \therefore \left.\dfrac{dy}{dx}\right|_{t=-1} = \dfrac{6}{2} = 3$ //

ⓑ at $t = 1$, $\quad x'(1) = 2$, $y'(1) = -6$ $\quad \therefore \left.\dfrac{dy}{dx}\right|_{t=1} = \dfrac{-6}{2} = -3$ //

ⓒ at $(0, 4) \Rightarrow t^3 - t = 0 \Rightarrow t = 0, \pm 1$

already discussed

$x'(0) = -1$
$y'(0) = 0$ $\quad \Rightarrow \dfrac{dy}{dx} = \dfrac{0}{-1} = 0$ (horizontal tangent)

(6) $\begin{cases} x = \cos 2t \\ y = \sin 3t \end{cases} \Rightarrow \begin{cases} x' = -2\sin 2t \\ y' = 3\cos 3t \end{cases}$

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ⓐ at $t = \pi/2$, $\quad x' = -2\sin\pi = 0$ (ouch!)
$y' = 3\cos 3\pi/2 = 0$

$\therefore$ do L'Hospital: $\left.\dfrac{dy}{dx}\right|_{t=\pi/2} = \lim_{t=\pi/2} \dfrac{-2\sin 2t}{3\cos 3t}$

$= \lim_{t=\pi/2} \dfrac{-4\cos 2t}{-9\sin 3t} = \dfrac{4}{9}$ //

ⓑ at $t = 3\pi/2$ $\quad x' = -2\sin 3\pi = 0$ (ouch!)
$y' = 3\cos(9\pi/2) = 3\cos(\pi/2) = 0$.

$\therefore$ do L'Hoprtal: $\left.\dfrac{dy}{dx}\right|_{t=3\pi/2} = \lim_{t\to 3\pi/2} \dfrac{-2\sin 2t}{3\cos 3t}$

$= \lim_{t\to 3\pi/2} \dfrac{-4\cos 2t}{9\sin 3t}$

$= \dfrac{4}{-9}$ //

(6) © at $(1,0)$. $\cos 2t = 1 \Rightarrow 2t = 2n\pi$
$\sin 3t = 0 \Rightarrow 3t = n\pi$

$\therefore$ for example, curve pass thru. $(1,0)$ @ $t=0$.
$x'(0) = 0$
$y'(0) = 3 \Rightarrow$ vertical tangent line //

(d) 4

$$\begin{cases} x = \cos 2t \\ y = \sin 7t \end{cases} \Rightarrow \begin{cases} x' = -2\sin 2t \\ y' = 7\cos 7t \end{cases}$$

$x' = 0$ when $\sin 2t = 0 \Rightarrow t = n\pi/2$, n any integer.
at those points: $y'\left(\frac{n\pi}{2}\right) = 7\cos\frac{7n\pi}{2} = \begin{cases} = 0, & \text{if } n \text{ odd} \\ \neq 0, & \text{if } n \text{ even} \end{cases}$

$\therefore$ vertical tangent @ $n\pi/2$, with n even,
that is, $t = n\pi/2 = k\pi$, with k integer.
$x(k\pi) = \cos 2k\pi = 1$
$y(7k\pi) = \sin(7k\pi) = (-1)^k \Rightarrow$ $\boxed{\text{vertical tangent line @ } (1,1) \ \& \ (1,-1)}$

when n is odd, $t = (2k+1)\pi/2$.

$\cos(2t) = \cos(2k+1)\pi = -1$

$\sin(7t) = \sin\left(\frac{\underbrace{7(2k+1)}_{\text{odd}}}{2}\pi\right) = \pm 1$

and $\frac{dy}{dx}$ to be computed by L'Hopital:

$$\lim_{t\to\frac{(2k+1)\pi}{2}} \frac{-2\sin 2t}{7\cos 7t} = \lim_{t\to\frac{(2k+1)\pi}{2}} \frac{-4\cos(2t)}{-49\sin(7t)} \neq 0 \ \& \text{ not } \infty.$$

$\therefore$ not vertical / horizontal.

For horizontal tangent lines: $y' = 0$
$\cos 7t = 0 \Rightarrow 7t = \frac{(2k+1)\pi}{2} \Rightarrow t = \frac{(2k+1)\pi}{14}$

and $x'\left(\frac{(2k+1)\pi}{14}\right) = -2\sin\left(\frac{(2k+1)}{7}\pi\right) \neq 0$ if $k \neq 3 + 7n$.

$\therefore$ location of horizontal tangent: will occur at these t.

(14) $\begin{cases} x = 2\cos 2t + \sin t \\ y = 2\sin 2t + \cos t \end{cases} \Rightarrow \begin{cases} x' = -4\sin 2t + \cos t \\ y' = 4\cos 2t - \sin t \end{cases}$

$x' = 0$: $-4\sin t\cos t + \cos t = \cos t(1 - 4\sin t) = 0$

$\cos t = 0$ OR $\sin t = 1/4$

$\cos t = 0$: $\quad y' = 4(2\cos^2 t + 1) - \sin t$ $\Rightarrow$ vertical tangent

$\Downarrow \qquad = 4 \pm 1 \neq 0.$

$\sin t = \pm 1$

$\sin t = 1/4$ $\quad y' = 4(2\cos^2 t + 1) - \sin t$ $\Rightarrow$ vertical tangent.

$\Downarrow \qquad = 4\left(\frac{15}{8} + 1\right) - \frac{1}{4} \neq 0$

$\cos^2 t = 1 - \sin^2 t = 15/16$

$\therefore$ Vertical tangent lines @ ...

$\cos t = 0 \Rightarrow \cos 2t = 1$ & $\sin t = \pm 1$, $\sin 2t = 0$

$\therefore (x, y) = (3, 0)$ OR $(1, 0)$

$\sin t = \frac{1}{4} \Rightarrow \cos 2t = 1 - \sin^2 t = 15/16.$

$\cos t = \pm\sqrt{1 - \frac{1}{16}} = \pm\sqrt{15}/4$ $\rightsquigarrow$ & plug into eq of x & y

$\sin 2t = 2\sin t\cos t = \pm\sqrt{15}/8$

Horizontal Tangent:

$y' = 0 \Rightarrow 4\cos 2t - \sin t = 0$

$\qquad = 4(1 - 2\quad) - \sin t$

$\qquad = 4 - 8\sin^2 t - \sin t$

$\Rightarrow 8\sin^2 t + \sin t - 4 = 0$

$\sin t = \frac{-1 \pm \sqrt{129}}{16} \Rightarrow \cos^2 t = 1 - \sin^2 t = \frac{4 + \left(\frac{-1 \pm \sqrt{129}}{16}\right)}{8}$

$\cos 2t = 2\cos^2 t + 1$, $\sin 2t = 2\sin t\cos t \ldots$

plug into original x-y equations to get location of horizontal tangent lines.

(18) $\begin{cases} x = 40t+5 \\ y = 20+3t-16t^2 \end{cases} \Rightarrow \begin{cases} x' = 40 \\ y' = 3-32t \end{cases} \Rightarrow$ Speed $= \sqrt{40^2+(3-32t)^2}$

ⓐ at $t=0$,
Speed $= \sqrt{40^2+3^2} = \sqrt{1609}$ //

ⓑ at $t=2$
Speed $= \sqrt{40^2+61^2} = \sqrt{5321}$ //

(22) $\begin{cases} x = 6\cos t \\ y = 2\sin t \end{cases} \Rightarrow \begin{cases} x' = -6\sin t \\ y' = 2\cos t \end{cases}$

Let t goes from 0 to 2π, then the loop has traveled exactly once, counterclockwise

$$A = -\int_0^{2\pi} 2\sin t\,(-6\sin t)\,dt$$
$$= 12\int_0^{2\pi} \sin^2 t\,dt$$
$$= 6\int_0^{2\pi}(1-\cos 2t)\,dt = 6\left(t - \frac{\sin 2t}{2}\right)\Big|_0^{2\pi}$$
$$= 12\pi \quad //$$

(26) $$A = -\int_{-\pi/2}^{\pi/2} t\cos t\,(t\sin t)'\,dt$$
$$= -\int_{-\pi/2}^{\pi/2} t\cos t\,(\sin t + t\cos t)\,dt$$
$$= -\int_{-\pi/2}^{\pi/2} t\cos t\sin t\,dt + \int_{-\pi/2}^{\pi/2} t\cos^2 t\,dt$$

this is all you need to do!

(30) $\begin{cases} x = 6\cos t + 5\cos 3t \\ y = 6\sin t - 5\sin 3t \end{cases} \Rightarrow \begin{cases} x' = -6\sin t - 15\sin 3t \\ y' = 6\cos t - 15\cos 3t \end{cases}$

$$\text{Speed} = \sqrt{(x')^2 + (y')^2} = \sqrt{36 + 225 + 180\sin t \sin 3t - 180\cos t\cos 3t}$$

as curve crosses x-axis, $y = 0$:

$6\sin t - 5\sin 3t = 0 \Rightarrow t = 0, \pi$

$\sin t = 0.$ $\quad \cos t\cos 3t = 1$

$\therefore$ Speed $= \sqrt{36 + 225 - 180} = \sqrt{81} = 9$ //.