HW4 Solutions (Optional)

§6.6. 42, 46, 55, 59.

(42) since $0 \le \cos^2 x \le 1 \Rightarrow 1 \le \sec^2 x < \infty$

⟨since $\sec^2 x = 1/\cos^2 x$⟩

$\therefore \dfrac{2+\sec^2 x}{x} \geq \dfrac{2}{x}$, and

$$\int_1^\infty \frac{2+\sec^2 x}{x}\,dx \geq \int_1^\infty \frac{2}{x}\,dx = 2\int_1^\infty \frac{1}{x}\,dx = \infty \quad (p=1).$$

$\therefore$ original integral diverges.

(46) Note: $\ln x \le x$ when $x \geq 1$

Reason: let $g(x) = \ln x - x$

$g(1) = \ln 1 - 1 = 0 - 1 < 0$

and $g'(x) = 1/x - 1 < 0$ when $x \geq 1$

that is, $g(x)$ is a decreasing function when $x \geq 1 \Rightarrow g(x) \le 0$ for $x \geq 1$, that means $\ln x \le x$ when $x \geq 1$.

Given that, we have

$$\frac{\ln x}{e^x+1} \le \frac{\ln x}{e^x} \le \frac{x}{e^x} \quad \text{when } x \geq 1$$

and $\int_1^\infty x/e^x\,dx < \infty$, using same method as prob. 50.

$$\therefore \int_1^\infty \frac{\ln x}{e^x+1}\,dx < \infty. \quad //.$$

(55) False! let $f(x) = 1/\sqrt{x} \to \infty$ as $x \to 0^+$

but $\int_0^1 \frac{1}{\sqrt{x}}\,dx < \infty$, since "$p$" $= \frac{1}{2} < 1$ //

(59) let $x = u/\sqrt{k}$, $dx = \frac{1}{\sqrt{k}}\,du$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-kx^2}\,dx = \frac{1}{\sqrt{k}}\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\frac{\pi}{k}} \checkmark.$$

§8.1 : 8, 15, 25, 28.

(8) clearly, $\lim_{n\to\infty} a_n = 2$. to prove that 2 is actually a limit,
let $\varepsilon > 0$,

$$|a_n - 2| = \left|\frac{2n+1}{n} - 2\right|$$
$$= \left|\frac{2n+1-2n}{n}\right| = \frac{1}{n}$$

$\therefore$ take $N = 1/\varepsilon$,
then if $n > N$, we have

$$|a_n - 2| = \frac{1}{n} < \varepsilon \quad , \text{ since } n > N = \frac{1}{\varepsilon}. \,//$$

(15) $a_n = (-1)^n \dfrac{n+2}{3n+2}$

if $n = 1, 3, 5, 7, \ldots, 2k+1$: odd integers
$(-1)^n = -1$ for all odd n.
and $\dfrac{n+2}{3n+2} \to \dfrac{1}{3}$. $\therefore \lim_{n\to\infty} a_n = -1/3$ when $n \to \infty$ via odd integers

On the other hand, if
$n = 2, 4, 6, 8, \ldots, 2k, \ldots$ even integers,
$(-1)^n = 1$. , yet $\dfrac{n+2}{3n+2} \to \dfrac{1}{3}$ still...

$\therefore \lim_{n\to\infty} a_n = \dfrac{1}{3}$

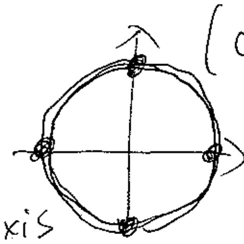
We've obtained two subsequences approaching different limits. $\therefore$ limit DNE.

(25) Say $n > 2$, we see

$$a_n = \frac{n!}{2^n} = \frac{n(n-1)\cdots 2\cdot 1}{2\cdot 2 \cdots 2\cdot 2} = \frac{1}{2}\left(\frac{n}{2}\right)\left(\frac{n-1}{2}\right)\cdots\left(\frac{2}{2}\right)$$

for ALL n
and if $n \to \infty$, $a_n \to \infty$.
diverges.

(28) $\left| \cos\frac{n\pi}{2} + \sin\frac{n\pi}{2} \right| = 1$, for ALL n



$\left(\cos\frac{n\pi}{2}, \sin\frac{n\pi}{2}\right)$

$\frac{n\pi}{2}$ = "$0°, 90°, 180°$, OR $270°$"

Since at $\theta = \frac{n\pi}{2}$, it is the angle where unit circle crosses coordinate axis (x OR y) ∵ see figure

∴ $\left(\cos\frac{n\pi}{2}, \sin\frac{n\pi}{2}\right)$ is one of the followings:

$(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$

$\Rightarrow \left| \cos\frac{n\pi}{2} + \sin\frac{n\pi}{2} \right| = 1 \quad \forall n.$

Theorfore, $a_n \to 2$, since $\frac{2n-1}{n+2} \to 2$. //