HW5# Solutions (Optional)

§8.1 : 26, 32, 56

(26) $a_n = \sqrt{n^2+n} - n$

$$= \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n}$$

$$= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n}$$

$$\overset{\div n \text{ top \& bottom}}{=} \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$\therefore$ clearly, $\lim_{n\to\infty} a_n = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$ //.

(32) Note: for $n \geq 3$, $1 \leq \ln n \leq n$ ← mentioned before

since $n \geq 3 \geq e$, $\ln 3 \geq \ln e = 1$

and clearly $-1 \leq (-1)^n \leq 1$.

$$\therefore \quad \frac{-n}{n^2} \leq (-1)^n \frac{\ln n}{n^2} \leq \frac{n}{n^2}$$

$$\frac{-n}{n^2} = \frac{-1}{n}, \qquad \frac{n}{n^2} = \frac{1}{n}$$

and since $\frac{1}{n}$ & $\frac{-1}{n}$ both go to 0, we conclude, by squeeze theorem, that $(-1)^n \frac{\ln n}{n^2} \to 0$.

(56) $p^n \to 0$ if $|p|<1$, ie. $-1<p<1$

$\therefore \frac{1}{p^n}$ diverges, if $|p|<1$

$p^n \to \pm\infty$ if $|p|>1$, ie. $p>1$ OR $p<1$

$\therefore \frac{1}{p^n} \to 0$.

$p=1$: $p^n=1$ for all n, $\frac{1}{p^n}=1$ always

$p=-1$, $(-1)^n = \begin{cases} 1; & n \text{ even} \\ -1; & n \text{ odd} \end{cases}$ $\frac{1}{(-1)^n} = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd}, \end{cases}$ not convergent

$\therefore \frac{1}{p^n}$ converges when $p<-1$ & $p \geq 1$

§8.2 : 7, 8, 13, 22.

(7) $\dfrac{4}{k(k+2)} = \dfrac{A}{k} + \dfrac{B}{k+2}$

$A(k+2) + Bk = 4$ $\qquad \dfrac{4}{k(k+2)} = \dfrac{2}{k} - \dfrac{2}{k+2}$

$\Rightarrow \begin{cases} A = 2 \\ B = -2. \end{cases}$

$\displaystyle\sum_{k=1}^{\infty} \frac{4}{k(k+2)} = 2 \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+2}\right) \quad \cdots (*)$

now. $\displaystyle\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right)$

$\displaystyle + \cdots + \left(\cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n-1}} + \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)$

$\displaystyle = 1 + \frac{1}{2} + \frac{1}{n+1} - \frac{1}{n+2} = \frac{3}{2} + \frac{1}{n+1} - \frac{1}{n+2}$

let $n \to \infty$, we see

$$\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{3}{2}$$

$\therefore \quad (*) = 3 \quad //.$

(8) $\displaystyle\lim_{k\to\infty} \frac{4k}{k+2} = 4 \neq 0 \quad \therefore \sum_{k=1}^{\infty} \frac{4k}{k+2}$ diverges <k-TH term tests>

(13) $\displaystyle\frac{2k+1}{k^2(k+1)^2} = \frac{k^2+2k+1-k^2}{k^2(k+1)^2}$

$\displaystyle = \frac{(k+1)^2-k^2}{k^2(k+1)^2} = \frac{1}{k^2} - \frac{1}{(k+1)^2}$

$\displaystyle\therefore \sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = \sum_{k=1}^{n}\left[\frac{1}{k^2} - \frac{1}{(k+1)^2}\right]$

$\displaystyle = \left(\frac{1}{1^2} - \cancel{\frac{1}{2^2}}\right) + \left(\cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}\right) + \left(\cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}}\right) + \cdots + \left(\cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}\right)$

$\displaystyle = 1 - \frac{1}{(n+1)^2}$

as $n \to \infty$, RHS $\to 1$. //.

(22) $\lim_{k\to\infty} (-1)^k \frac{k^3}{k^2+1} \neq 0$

since letting k be even integer and goes to ∞ via $2, 4, 6, 8, \dots$

$$(-1)^k \frac{k^3}{k^2+1} \to +\infty.$$

$\therefore$ By k-th test, series diverges//