HW#6 Solutions (Graded)

§8.2: 18, 28

(18) $\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1-\frac{1}{2}} = 2$ (Geometric Series w/ $a=1$, $r=\frac{1}{2}$)

$\sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$ (Geometric Series w/ $a=1$, $r=\frac{1}{3}$)

Since both series converge, we may distribute Σ:

$$\sum_{k=0}^{\infty}\left(\frac{1}{2^k} - \frac{1}{3^k}\right) = \sum_{k=0}^{\infty} \frac{1}{2^k} - \sum_{k=0}^{\infty} \frac{1}{3^k} = 2 - \frac{3}{2} = \frac{1}{2} \;//$$

(28) If for some m, $\sum_{k=m}^{\infty} a_k = L < \infty$, converges

then $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{m-1} a_k + \sum_{k=m}^{\infty} a_k$

$= \sum_{k=1}^{m-1} a_k + L$

but $\sum_{k=1}^{m-1} a_k$ is a finite number. ‹finitely many terms›

$\therefore$ it makes $\sum_{k=1}^{\infty} a_k$ converges to a finite number

and it doesn't diverge. We have proved the equivalent statement of the problem (If A then B = not B then not A)

§8.3: 4, 6, 18, 21, 26, 34, 36, 44, 56.

(4) $\sum_{k=6}^{\infty} \frac{4}{\sqrt{k}} = 4 \sum_{k=6}^{\infty} \frac{1}{k^{1/2}} = \infty$ (p series w/ $p = \frac{1}{2} < 1$)

(6) $a_k = \frac{k^2+1}{k^3+3k+2} \sim \frac{k^2}{k^3} = \frac{1}{k}$ when k is large

$\therefore$ compare a_k to $b_k = 1/k$

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^3+k}{k^3+3k+2} = 1 > 0$

$\therefore$ may apply LCT.

note that $\sum b_k = \sum \frac{1}{k} = \infty$ (harmonic series)

$\therefore$ $\sum a_k$ also diverges//

(18) $\frac{2}{\sqrt{k^2+4}} \sim \frac{2}{k}$ when k is large

$\therefore$ let $b_k = \frac{1}{k}$.

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{2k}{\sqrt{k^2+4}} = 2 > 0$

$\therefore$ may apply LCT

and clearly $\sum b_k = \infty \Rightarrow \sum \frac{2}{\sqrt{k^2+4}} = \infty$ //

(21) let $f(x) = \frac{\tan^{-1}x}{1+x^2}$

continuous ✓ $(1+x^2 \neq 0)$

≥ 0 sure... since $\tan^{-1}x \geq 0 \ \forall x \geq 0$

decreasing: $f'(x) = \frac{1-2x\tan^{-1}x}{(1+x^2)^2} \leq 0$ if x is large enough

{since $\tan^{-1}x \xrightarrow{x\to\infty} \frac{\pi}{2} > 0$}

$\therefore$ $f(x)$ is decreasing when x is large enough. $\therefore$ may apply integral test...

$$\int_1^\infty \frac{\tan^{-1}x}{1+x^2}\,dx$$

$$= \int_{\pi/4}^{\pi/2} u\,du \qquad u = \tan^{-1}x,\quad dx = (1+x^2)\,du$$

$$= \frac{u^2}{2}\Big|_{\pi/4}^{\pi/2} < \infty$$

$$\therefore \sum_{k=1}^{\infty} \frac{\tan^{-1}k}{1+k^2} \text{ converges}_{//}$$

(26) $-1 \le \cos k \le 1$

$\therefore \quad 1 \le 2 + \cos k \le 3$

OR $\dfrac{1}{k} \le \dfrac{2+\cos k}{k} \le \dfrac{3}{k}$

and since $\Sigma 1/k = \infty$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{2+\cos k}{k} = \infty_{//}$$

(34) note that $(k+1)\sqrt{k} + k^2\sqrt{k+1}$

$= k^{3/2} + k^{1/2} + \sqrt{k^5+1}$

$\sim k^{3/2} + k^{1/2} + k^{5/2}$ when k is big

$\sim k^{5/2}$

$\therefore$ compare with $b_k = k/k^{5/2} = 1/k^{3/2}$

$$\lim_{k\to\infty} \frac{\dfrac{2k+1}{(k+1)\sqrt{k}+k^2\sqrt{k+1}}}{1/k^{3/2}} = \lim_{k\to\infty} \frac{2k^{5/2}+k^{3/2}}{(k+1)\sqrt{k}+k^2\sqrt{k+1}}$$

$/k^{5/2}$

$$= \lim_{k\to\infty} \frac{2+1/k}{(1+1/k)\sqrt{1/k^2}+\sqrt{1+1/k}} = 2 > 0$$

$\therefore$ LCT applies, and since

$$\Sigma b_k = \Sigma \frac{1}{k^{3/2}} < \infty \quad \left(\frac{3}{2} > 1\right) \Rightarrow \Sigma \frac{2k+1}{(k+1)\sqrt{k}+k^2(k+1)} < \infty$$

//

(36) $a_k > 0$ and $\sum_1^\infty a_k$ converges, $\therefore$ $a_k \to 0$

and $\therefore$ there is some K s.t.
$0 < a_k < 1$ for ALL $k > K$.
and $\therefore 0 < a_k^2 \leq a_k$ " " " "

By comparison test
$$0 < \sum_{k=K}^\infty a_k^2 < \sum_{k=K}^\infty a_k < \infty$$
$$\left(\text{Since } \sum_{k=K}^\infty a_k < \sum_{k=1}^\infty a_k < \infty \right)$$

$$\sum_{k=K}^\infty a_k < \infty \Rightarrow \sum_{k=1}^\infty a_k = (a_1 + \cdots + a_{K-1}) + \sum_{k=K}^\infty a_k$$
$$< \infty. \quad \text{QED}$$

(44) when $p > 1$, $\lim_{k\to\infty} k^{p-1} e^{kp} = \infty \neq 0$
$\therefore$ series diverges

$0 < p < 1$, $\lim_{k\to\infty} k^{p-1} e^{kp} = \infty \neq 0$ $\therefore$ diverges.
⟨Do L'Hopital on $\lim_{x\to\infty} x^{p-1} e^{xp}$⟩

$p = 0$: $\sum_{k=1}^\infty \frac{1}{k}$ diverges

$p < 0$: we claim that
$e^{kp} \leq k^{-p-1}$ if k is large enough
$$\lim_{k\to\infty} \frac{e^{kp}}{k^{-p-1}} = 0$$
⟨using L'Hopital⟩

$\therefore$ $k^{p-1} e^{kp} \leq k^{p-1} k^{-p-1} = k^{-2} = \frac{1}{k^2}$ if k large enough.

$\therefore$ $\sum_{k=K}^\infty k^{p-1} e^{kp} \leq \sum_{k=K}^\infty \frac{1}{k^2} < \infty$ $\therefore$ converges

(15b) Given that $\sum_{k=1}^{\infty} a_k$ diverges $\left(\therefore = \infty\right)$ since $a_k > 0$ & $a_k, b_k > 0$

4 ⓐ if $b_k \geq a_k \quad \forall k \geq 10$

$$\geq \sum_{k=10}^{\infty} a_k = \sum_{k=1}^{\infty} a_k - \underbrace{(a_1 + \cdots + a_9)}_{<\infty} = \infty - (<\infty) = \infty$$

$\therefore \sum_{k=10}^{\infty} b_k = \infty$, & $\sum_{k=1}^{\infty} b_k \geq \sum_{k\geq 10}^{\infty} b_k = \infty$

$b_k > 0$ $\therefore$ diverges //

ⓑ can't tell !

say $a_k = \frac{1}{\sqrt{k}}$, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \infty$

let $b_k = \frac{1}{k^2}$, $\lim b_k/a_k = 0$, but $\sum b_k$ converges

let $b_k = \frac{1}{k}$, $\lim b_k/a_k = 0$, but $\sum b_k$ diverges

ⓒ can't tell.

let $a_k = \frac{1}{6k}$

let $a_k = \frac{1}{7k}$, then of course $b_k \leq a_k \quad \forall k \geq 6$ (actually $\forall k$)
& $\sum b_k$ diverges.

let $b_k = \frac{1}{k^2}$, $\therefore b_k \leq a_k \quad \forall k \geq 6$ & $\sum b_k$ converges.

ⓓ $\lim_{k\to\infty} \frac{b_k}{a_k} = \infty$, then $\exists K$ s.t. $\forall k \geq K$,
$\frac{b_k}{a_k} > 1$, (ie. $b_k > a_k$)

$\therefore \sum_{k=K}^{\infty} b_k > \sum_{k=K}^{\infty} a_k = \infty$ see ⓐ

$\therefore \sum_{k=K}^{\infty} b_k = \infty \Rightarrow \sum_{k=1}^{\infty} b_k = \infty$ //