HW 6 Solutions (Optional)

§8.2: 19, 27, 33

(19) Since both $\Sigma\, 2/3^k$ & $\Sigma\, 1/2^k$ converge
(geometric series with $r = 2/3$ & $1/2$, respectively)
We have

$$\sum_{k=2}^{\infty}\left(\frac{2}{3^k}+\frac{1}{2^k}\right) = \sum_{k=2}^{\infty}\frac{2}{3^k} + \sum_{k=2}^{\infty}\frac{1}{2^k}$$

$$= \frac{\frac{2}{9}}{1-\frac{1}{3}} + \frac{\frac{1}{4}}{1-\frac{1}{2}} = \frac{1}{3}+\frac{1}{2} = \frac{5}{6} \;/\!/$$

(27) $\sum_{k=1}^{\infty} a_k = L < \infty$

$$\sum_{k=1}^{m-1} a_k + \sum_{k=m}^{\infty} a_k = (a_1 + \cdots + a_{m-1}) + \sum_{k=m}^{\infty} a_k$$

$$\Rightarrow \sum_{k=m}^{\infty} a_k = L - (a_1 + \cdots + a_{m-1}) < \infty$$

(33) $0.\bar{9} = 0.9999\cdots$

$$= 0.9 + 0.09 + 0.009 + \cdots$$

$$= 0.9 + 0.9\cdot\frac{1}{10} + 0.9\,\frac{1}{10^2} + \cdots$$

$$= \sum_{k=1}^{\infty}(0.9)^k = \frac{0.9}{1-\frac{1}{10}} = \frac{0.9}{0.9} = 1 \;/\!/$$

§8.3 : 11, 22, 27, 29, 41, 42, 57.

(11) compare to $b_k = \frac{1}{k^2}$

$$\lim_{k\to\infty} \frac{\frac{2k}{k^3+1}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{2k^3}{k^3+1} = 2 > 0$$

$\therefore$ LCT applies ... & since $\sum b_k = \sum \frac{1}{k^2} < \infty$

$\Rightarrow \sum_{k=1}^{\infty} \frac{2k}{k^3+1} < \infty$ //

(22) $a_k = \frac{\sin^{-1}(1/k)}{k^2}$; when $k \geq 0$, $0 \leq \sin^{-1}(1/k) \leq \pi/2$

$$\therefore \quad 0 \leq \frac{\sin^{-1}\left(\frac{1}{k}\right)}{k^2} \leq \frac{\pi/2}{k^2}$$

& $\sum_{k=1}^{\infty} (\pi/2)/k^2 = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty \quad (p=2>1)$

$\therefore \sum_{k=1}^{\infty} \frac{\sin^{-1}(1/k)}{k^2} < \infty$, converges//

(27) compare $a_k = \frac{k^4+2k-1}{k^5+3k^2+1}$ to $b_k = \frac{1}{k}$

it's clear that ..

$$\lim_{k\to\infty} \frac{a_k}{b_k} = 1 > 0$$

$\therefore$ LCT applies, since $\sum \frac{1}{k} = \infty$,
we conclude that $\sum_{k=1}^{\infty} \frac{k^4+2k-1}{k^5+3k^2+1}$ diverges//

(29) $\lim_{k\to\infty} \frac{k+1}{k+2} = 1 \neq 0$ By kTH term test, series diverges. //

(41) let $f(x) = \frac{1}{x(\ln x)^p} \geq 0$. since $\ln x > 0 \ \forall x \geq 2$. continuous (x, $\ln x \neq 0$)

$$f'(x) = \frac{-[(\ln x)^p + p(\ln x)^{p-1}]}{x^2(\ln x)^{2p}}$$

$$= \frac{-[(\ln x)^{p-1}[\ln x + p]]}{x^2(\ln x)^{2p}} \leq 0 \quad \therefore \text{decreasing}.$$

$\therefore$ may apply integral test to $f(x)$ ...,

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx$$

$$= \int_{\ln 2}^\infty \frac{1}{u^p}\,du \qquad u = \ln x, \quad dx = x\,du$$

$$= \begin{cases} < \infty, & \text{if } p > 1 \\ = \infty, & \text{if } p \leq 1 \end{cases}$$

$\therefore$ series converges exactly when $p > 1$.

(42) let $f(x) = \frac{1}{(a+bx)^p}$; $a > 0$, $b > 0$.

$f(x)$ clearly ≥ 0, and continuous $(a+bx > 0)$

$f(x)$ is decreasing since $(a+bx)^p$ is increasing.

$$\int_0^\infty \frac{1}{(a+bx)^p}\,dx = \frac{1}{b}\int_a^\infty \frac{1}{u^p}\,du \qquad \begin{matrix} u = a+bx \\ dx = du/b \end{matrix}$$

converges exactly when $p > 1$, and so does the series.

(57) For all k, $\frac{1}{2k+1} \quad \frac{1}{2k+2} \quad 0$

& $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k+2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k+1}$

$= \frac{1}{2} \left(\frac{1}{2} + \frac{1}{3} + \cdots \right)$

$= \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots \right) - \frac{1}{2}$

$= \frac{1}{2} \underbrace{\sum_{k=1}^{\infty} \frac{1}{k}}_{= \infty} - \frac{1}{2} = \infty$

By comparison test, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k+1} = \infty$ //