HW#7 Solutions (Graded)

§8.4: 4, 6, 10, 16, 22, 44

(4) $(-1)^k \frac{k^2}{k+1} \not\to 0$ (actually limit doesn't exist)

$\therefore$ By k_{TH} term test, series diverges //

(6) clearly $\frac{2k-1}{k^3} \to 0$ as $k \to \infty$

✓ and let $f(x) = \frac{2x-1}{x^3}$

$$f'(x) = \frac{2x^3 - 3x^2(2x-1)}{x^6} = \frac{-5x^3+3x^2}{x^6}$$
$$= \frac{-5x+3}{x^4} < 0 \quad \text{if } x > 1$$

$\therefore$ f decreasing after $x > 1$.

$\Rightarrow a_k$ " ", $k > 1$

By A.S.T, series converges.

(10) let $f(x) = 3^x/x$

$$\lim_{x\to\infty} \frac{3^x}{x} = \lim_{x\to\infty} 3^x \cdot \ln 3 = \infty.$$

$\therefore$ $3^k/k \to \infty$, and $(-1)^k\, 3^k/k \not\to 0$.

By k_{TH} term test, series diverges //

(16) $\frac{a_{k+1}}{a_k} = \frac{\frac{(k+1)!}{3^{k+1}}}{\frac{k!}{3^k}} = \frac{3^k}{3^{k+1}} \frac{(k+1)!}{k!} = \frac{k+1}{3} > 1$ when $k > 2$

$\therefore$ a_k is not decreasing ... AST doesn't work.

But it doesn't mean series will diverge !!

TRY k_{TH} term test ...

Note that $\frac{a_{k+1}}{a_k} = \frac{k+1}{3} > 2$ when $k > 5$

$\therefore$ we have $a_k > 2^{k-5} a_5$ $\forall k > 5$, which goes

to ∞. $\therefore$ $(-1)^k a_k \not\to 0$ < actually limit doesn't exist .. by considering two subsequences: even terms and odd terms, one goes to $+\infty$, & the other goes to $-\infty$ >.
$\therefore$ $\Rightarrow$ series diverges //

(22) $a_k = \frac{1}{\ln k} \to 0$ since $\ln k \to \infty$
it's decreasing, since $\ln k$ is increasing
$\therefore$ AST $\Rightarrow$ series converges //

(44) $1/k^p \to 0$ exactly when $p > 0$.
($1/k^p \to \infty$ if $p < 0$, & $(-1)^k(1/k^p) \not\to 0$. kth term test
$\Rightarrow$ diverges.) $\therefore$

As for decreasing, we let

$$f(x) = \frac{1}{x^p} = x^{-p} \quad , p > 0$$

$$f'(x) = -p\,x^{-p-1} = \frac{-p}{x^{p+1}} < 0 \quad \forall x.$$

$\therefore$ AST applies ...
$\therefore$ $\sum (-1)^k \frac{1}{k^p}$ converges for all $p > 0$.

if $p < 0$, $\frac{1}{k^p} = k^{-p} \to \infty$ ($-p > 0$) as $k \to \infty$ $\therefore$ $(-1)^k \frac{1}{k^p} \not\to 0$.
$\therefore$ series diverges.

if $p = 0$, $\frac{1}{k^p} = 1$, and $\sum (-1)^k$ diverges.

$\therefore$ no other value of p makes the series converge

§8.5: 4, 12, 22, 26, 28, 38, 44

(4) the series
$\sum_{k=0}^{\infty} \frac{2}{3^k}$ is a geometric series converging to $\frac{2}{1-1/3} = 3$

$\therefore$ series absolutely converges //

(12) $C_k = (-1)^k \frac{3^k \cdot k^2}{2^k} \Rightarrow |C_k| = k^2 \left(\frac{3}{2}\right)^k \xrightarrow{k\to\infty} \infty$

(both parts $\to \infty$)

$\therefore (-1)^k \frac{3^k \cdot k^2}{2^k} \not\to 0$ (again consider two subseq's: odd & even terms)

KTH term test $\Rightarrow$ series diverges ...

(22) Use root test ...

$$\lim_{k\to\infty} |C_k|^{1/k} = \lim_{k\to\infty} \frac{e^k}{k^2} = \infty > 1$$

use L'Hopital

$\therefore$ series diverges //

(26) $\sin k\pi = 0$ for all k. integers

$\therefore \sum_{k=1}^{\infty} \frac{\sin k\pi}{k} = 0$ < Oh yeah !!! >

(28) $|C_k| = \frac{1}{k \ln k}$ $\quad \lim_{k\to\infty} \frac{|C_{k+1}|}{|C_k|} = 1$ .. oops...

instead, try integral test.

Let $f(x) = \frac{1}{x \ln x} \geq 0$ (for $x \geq 1$)

continuous for all $x > 0$.

$f'(x) = \frac{-[\ln x + 1]}{(x \ln x)^2} < 0$, for $x > e$.

$\therefore$ integral test is applicable, and

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \ln(\ln x) \Big|_2^{\infty} = \infty$$

$\therefore \sum |C_k|$ diverges.

However, AST says that the alternating series converges:

$\frac{1}{k \ln k} \to 0$ & $\frac{1}{k \ln k}$ decreasing

⟨easy to check⟩

∴ the series converges conditionally.

(38) $\lim_{k\to\infty} \frac{|C_{k+1}|}{|C_k|} = \lim_{k\to\infty} \frac{\frac{3^{k+1}}{(k+1)^2 4^{k+1}}}{\frac{3^k}{k^2 4^k}}$

$= \lim_{k\to\infty} \frac{3}{4} \frac{k^2}{(k+1)^2} = \frac{3}{4} < 1$

∴ Series converges absolutely //

(44) $\lim_{k\to\infty} \frac{|C_{k+1}|}{|C_k|} = \lim_{k\to\infty} |P| \frac{k^2}{(k+1)^2} = |P|$

∴ $< 1 \Rightarrow$ converges absolutely

$> 1 \Rightarrow$ diverges.

$|p| = 1$:

not the "p" in the problem

$p = 1$ $\sum \frac{1}{k^2} < \infty$ (p series w/ $p = 2$)

$p = -1$: $\sum \frac{(-1)^k}{k^2} < \infty$ (by AST test)

$\frac{1}{k^2} \to 0$ & decreasing

∴ Series converges when $|p| \le 1$.

i.e. $-1 \le p \le 1$.