HW 8 (Graded) Solutions

§8.3: 46, 49.

(46) $R_{100} \le \int_{100}^{\infty} \frac{4}{x^2}\,dx$

$= -\frac{4}{x}\Big|_{100}^{\infty} = \frac{4}{100}$

$\therefore$ error $\le \frac{1}{25}$ // ANS.

(49) $R_{40} \le \int_{40}^{\infty} x e^{-x^2}\,dx$

$= \frac{e^{-x^2}}{2}\Big|_{40}^{\infty} = \frac{e^{-160}}{2}$

$\therefore$ error $\le \frac{e^{-160}}{2}$ // ANS.

§8.4: 28, 32, 34.

(28) $\frac{(n+1)^2}{10^{n+1}} \le \frac{1}{100}$ $\therefore$ $n=2$ would suffice.

but our series here starts at $k=4$.
$\therefore$ just take the first term,
ie. $\frac{16}{10^4}$ //

(32) $\frac{3}{(n+1)^5} < \frac{1}{100}$, take $n=3$, ie. first term of our series,

$-\frac{3}{3^5} = -\frac{1}{81}$ //

(34) $\frac{2^{n+1}}{(n+1)!} < \frac{1}{10^4}$. take $n=9$.

$\frac{2^{10}}{10!} = \frac{\cancel{1024}\ 16}{10\cdot 9\cdot \cancel{8}\cdot 7\cdot 6\cdot 5\cdot \cancel{4}\cdot 3\cdot \cancel{2}}$

$= \frac{16}{18900} < \frac{1}{10^4}$

Since k start at 0, we need $9+1=\underline{\underline{10}}$ terms //

§ 8.6: 10, 14, 18, 20, 26, 31, 36.

(10) $\sum_{k=0}^{\infty} (x-3)^k$ ← geometric series w/ $r = x-3$.

$= \dfrac{1}{1-(x-3)} = \dfrac{1}{4-x}$, converges exactly when $|x-3| < 1$, ie. $2 < x < 4$

(14) $\sum_{k=0}^{\infty} 3\left(\frac{x}{4}\right)^k$

$= 3\sum_{k=0}^{\infty}\left(\frac{x}{4}\right)^k$ ← geom. series w/ $r = \frac{x}{4}$. Converges exactly when $\left|\frac{x}{4}\right| < 1$ ie $-4 < x < 4$

$= 3 \dfrac{1}{1-\frac{x}{4}}$

$= \dfrac{12}{4-x}$ //

(18) $\sum_{k=0}^{\infty} \frac{k}{2^k} x^k$

Ratio test: $L(x) = \lim_{k\to\infty} \dfrac{\frac{k+1}{2^{k+1}} |x|^{k+1}}{\frac{k}{2^k} |x|^k}$

$= \lim_{k\to\infty} \frac{1}{2}\frac{k+1}{k}|x| = \frac{|x|}{2}$

$L(x) < 1$ when $-2 < x < 2$

$x = 2$: $\sum_{k=0}^{\infty} k = \infty$. $x = -2$. $\sum_{k=0}^{\infty} (-1)^k k$ diverges ($(-1)^k k \not\to 0$)

∴ interval of convergence $= (-2, 2)$.
radius " " $= 1$.

(20) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \cdot 4^k}(x+2)^k$

$$L(x) = \lim_{k\to\infty} \frac{\frac{1}{(k+1)4^{k+1}}|x+2|^{k+1}}{\frac{1}{k\cdot 4^k}|x+2|^k}$$

$$= \lim_{k\to\infty} \frac{k}{4(k+1)}|x+2| = \frac{|x+2|}{4}$$

$L(x)<1$: $\frac{|x+2|}{4} < 1 \iff |x+2| < 4$

$\iff -6 < x < 2$

$x = 2$: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} < \infty$ (alternating harmonic series)

$x = -6$ $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \cdot 4^k}(-4)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k \cdot \cancel{4^k}}{k \cdot \cancel{4^k}}$

$(-1)^{2k+1} = -1$ odd

$= -\sum_{k=1}^{\infty} \frac{1}{k}$ diverges (harmonic series)

$\therefore$ interval of convergence $[-6, 2]$.

radius " " $= \frac{1}{2}(2+6) = 4$. //

(26) $\sum_{k=2}^{\infty} \frac{(k!)^2}{(2k)!} x^k$

$$L(x) = \lim_{k\to\infty} \frac{\frac{((k+1)!)^2}{(2k+2)!}|x|^{k+1}}{\frac{(k!)^2}{(2k)!}|x|^k}$$

$$= \lim_{k\to\infty} \left(\frac{(k+1)!}{k!}\right)^2 \frac{(2k)!}{(2k+2)!}|x|$$

$$= \lim_{k\to\infty} (k+1)^2 \frac{1}{(2k+2)(2k+1)}|x| = \frac{|x|}{4}.$$

$\frac{|x|}{4} < 1$ when $-4 < x < 4$

$x = 4$: $\sum_{k=2}^{\infty} \frac{(k!)^2}{(2k)!} \cdot 4^k$

$$a_2 = \frac{4}{4!} \cdot 16 > 0$$

and $$\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(2k+2)(2k+1)} \cdot 4$$

$$= \frac{k^2+2k+1}{4k^2+6k+2} \cdot 4$$

$$\frac{k^2+2k+1}{4k^2+8k+4} \cdot 4 = \frac{1}{4} \cdot 4 = 1$$

$\therefore$ a_k is an increasing sequence

and since a , $a_k \not\to 0$. $\therefore$ diverges by k-TH term test.

for $x = -4$. $\sum_{k=2}^{\infty} (-1)^k \frac{(k!)^2}{(2k)!} \cdot 4^k$

again, $(-1)^k \frac{(k!)^2}{(2k)!} \cdot 4^k \not\to 0$

(since there is a subsequence, the even part (all even k) that doesn't approach 0)

$\therefore$ interval of convergence $= (-4, 4)$
radius " " $= 4$

(31) $f(x) = 3\tan^{-1} x = 3\int \frac{1}{1+x^2}\,dx$

$= 3\int \sum_{k=0}^{\infty} (-x^2)^k\,dx$; $|x^2| < 1$, or $-1 < x < 1$

$= 3\sum_{k=0}^{\infty} \int (-1)^k x^{2k}\,dx$

$= 3\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$, $-1 < x < 1$

radius $= 1$ //

(36) $f(x) = \ln(4+x) = \int \frac{1}{4+x}\,dx$

$= \frac{1}{4}\int \frac{1}{1+\frac{x}{4}}\,dx$

$= \frac{1}{4}\int \sum_{k=0}^{\infty} \left(-\frac{x}{4}\right)^k dx$, $\left|\frac{x}{4}\right| < 1$

i.e. $-4 < x < 4$

$= \frac{1}{4}\int \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{4^k}\,dx$

$= \frac{1}{4}\sum_{k=0}^{\infty} \int (-1)^k \frac{x^k}{4^k}\,dx$

$= \frac{1}{4}\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{4^k(k+1)}$, $-4 < x < 4$

radius $= 4$ //