HW 9 (Graded) Solutions.

§8.6 : 38

(38) $\sum_{k=1}^{\infty} \frac{\cos(x/k)}{k}$

for any x, we see that

$$\lim_{k\to\infty} \cos\left(\frac{x}{k}\right) = 1$$

$\therefore \exists N$ large enough, s.t. $\frac{1}{2} < \cos(\frac{x}{k}) < \frac{3}{2}$.

$$\sum_{k=N}^{\infty} \frac{\cos(\frac{x}{k})}{k} > \frac{1}{2} \sum_{k=N}^{\infty} \frac{1}{k} = \infty \quad \text{(harmonic series)}$$

$$\therefore \sum_{k=N}^{\infty} \frac{\cos(\frac{x}{k})}{k} = \infty.$$

and $\therefore \sum_{k=1}^{\infty} \frac{\cos(\frac{x}{k})}{k}$ diverges for ALL x.

§8.7 : 1, 14, 30, 34, 36, 40. Additional.

(1) $f(x) = \cos x$, $\quad c = 0$

$$\begin{aligned}
f^{(0)}(x) &= \cos x, & f^{(0)}(0) &= 1. \\
f^{(1)}(x) &= -\sin x, & f^{(1)}(0) &= 0 \\
f^{(2)}(x) &= -\cos x, & f^{(2)}(0) &= -1 \\
f^{(3)}(x) &= \sin x, & f^{(3)}(0) &= 0 \\
f^{(4)}(x) &= \cos x, & f^{(4)}(0) &= 1 \\
&\vdots
\end{aligned}$$

$\therefore f^{(k)}(0) = 0$ for all odd k.

for even k, we see

$$f^{(2k)}(0) = \begin{cases} 1, & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$$

$\therefore$ Taylor (Maclaurin) Series of $\sin x$ is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$

(~~14~~) $\cos x^2 \text{ "=" } \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^4}{2} + \frac{x^6}{24} - \frac{x^8}{720} + \cdots$

$$\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \cos x^2 \, dx \approx \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \left(1 - \frac{x^4}{2} + \frac{x^6}{24} - \frac{x^8}{720}\right) dx$$

$$= \left(x - \frac{x^5}{10} + \frac{x^7}{168} - \frac{x^9}{6480}\right)\Bigg|_{-\sqrt{\pi}}^{\sqrt{\pi}} = \cdots$$

(14) (~~30~~)

$$f^{(0)}(x) = \frac{1}{x}$$

$$f^{(1)}(x) = -\frac{1}{x^2}$$

$$f^{(2)}(x) = \frac{2}{x^3} = \frac{2!}{x^3}$$

$$f^{(3)}(x) = \frac{-3 \cdot 2!}{x^4} = \frac{-3!}{x^4}$$

$$f^{(4)}(x) = \frac{4!}{x^5}$$

$$\vdots$$

$$f^{(k)}(x) = \frac{(-1)^k k!}{x^{k+1}} \quad ; \quad f^{(k)}(-1) = \frac{(-1)^k \cdot k!}{(-1)^{k+1}} = -k!$$

$\therefore$ Taylor series of $\frac{1}{x}$ at $c=-1$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(-1)}{k!} (x+1)^k = -\sum_{k=0}^{\infty} (x+1)^k .$$

a geometric series w/ $r = (x+1)$

$\therefore$ converges exactly when $|x+1| < 1$.

OR $-2 < x < 0$ //

(30) $\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{(2k+1)!} = \sin \pi = 0$ //

(34) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$

$\therefore\ e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$

$\frac{e^x - 1}{x} = 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \cdots$

$= \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}$ //

(36) $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

$\therefore\ \sin x^2 = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!}$ //

(40) Recall fundamental theorem of calculus:

$F(x) = \int_a^x g(u)\, du \Rightarrow F'(x) = g(x).$

$\therefore$ Here, $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$

$\mathrm{erf}^{(1)}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \Rightarrow \mathrm{erf}^{(1)}(0) = \frac{2}{\sqrt{\pi}}$

$\mathrm{erf}^{(2)}(x) = \frac{-4}{\sqrt{\pi}} x e^{-x^2} \Rightarrow \mathrm{erf}^{(2)}(0) = 0$

$\mathrm{erf}^{(3)}(x) = \frac{-4}{\sqrt{\pi}} \left(e^{-x^2} - 2x^2 e^{-x^2}\right) \Rightarrow \mathrm{erf}^{(3)}(0) = -\frac{4}{\sqrt{\pi}}$

$\mathrm{erf}^{(4)}(x) = \frac{-4}{\sqrt{\pi}} \left(-2x e^{-x^2} - 4\underbrace{(x(\cdots))}_{x=0 \Rightarrow 0}\right) \Rightarrow \mathrm{erf}^{(4)}(0) = 0$

and 4TH order Taylor series

$= \mathrm{erf}^{(0)}(0) + \mathrm{erf}^{(1)}(0)\, x + \frac{\mathrm{erf}^{(2)}(0)}{2} x^2 + \frac{\mathrm{erf}^{(3)}(0)}{6} x^3 + \frac{\mathrm{erf}^{(4)}(0)}{24} x^4$

Additional:

Given $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

Since it converges everywhere, we may apply term by term differentiation:

$$\cos x = \frac{d}{dx} \sin x$$

$$= \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{d}{dx}\left[\frac{(-1)^k}{(2k+1)!} x^{2k+1}\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (2k+1) x^{2k}$$

$$\frac{n}{n!} = \frac{1}{(n-1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

✓

§8.8 : 8, 11

(8) $\sin x^2 = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!} = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \cdots$ ⟨ see prob. (36) of 8-6 ⟩

$$\lim_{x\to 0} \frac{\sin x^2 - x^2}{x^6} = \lim_{x\to 0} \frac{-\frac{x^6}{6} + \frac{x^{10}}{120} - \cdots}{x^6}$$

$$= \lim_{x\to 0} -\frac{1}{6} + \frac{x^4}{120} + \frac{x^8}{\cdots} - \cdots$$

$$= -\frac{1}{6} \quad //$$

(11) $\lim_{x\to 0} \frac{e^x - 1}{x}$

$= \lim_{x\to 0} (1 + \frac{x}{2} + \frac{x^2}{6} + \cdots)$ ⟨ see prob. (34) of 8-8 ⟩

$= 1 \quad //$