

A new type of edge-derived vertex coloring

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Abstract

We study the minimum number of weights assigned to the edges of a graph G with no component K_2 so that any two adjacent vertices have distinct sets of weights on their incident edges. The best possible upper bound on this parameter is proved.

Keywords: Edge coloring; Chromatic number; Edge-weighting

1 Introduction

All graphs we discuss are simple (note that most results hold for multigraphs too) and finite. Let G be a graph and k a non-negative integer. A k -edge-weighting of G is a mapping $\varphi : E(G) \rightarrow \{1, 2, \dots, k\}$. (In this paper the term “weighting” will always refer to edges while “coloring” will always refer to vertices.) The *weight set* (with respect to φ) of a vertex $x \in V(G)$ is the set $S_\varphi(x)$ of weights on edges incident to x (the subscript φ can be omitted when it does not cause confusion). A k -edge-weighting φ is called *vertex-coloring by sets* if $S_\varphi(x) \neq S_\varphi(y)$ whenever vertices x, y are adjacent (typically we will

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omit the phrase “by sets”). For a graph G we are interested in the minimum k such that there exists a k -edge-weighting of G that is vertex-coloring. If G has a component K_2 , then G cannot have a vertex-coloring edge-weighting, so we (have to) assume that G has no such component. If G is a graph with components G_1, \dots, G_n , then we can take the maximum of these minima componentwise, so the analysis of vertex-coloring edge-weightings can be restricted to connected graphs. Therefore, all graphs will be assumed to be connected unless otherwise stated.

In this paper we will study k -edge-weightings that are vertex-coloring by sets. Actually, different kinds of edge-weightings deriving proper vertex-colorings have been studied. We say that a k -edge-weighting of G is *vertex-coloring by sums* if for every edge xy in G , the sum of the weights appearing on the edges incident to x is distinct from the sum of the weights appearing on the edges incident to y . Similarly, we say that a k -edge-weighting of G is *vertex-coloring by multisets* if for every edge xy in G , the multiset of the weights appearing on the edges incident to x is distinct from the multiset of the weights appearing on the edges incident to y . Considering these related concepts without appropriate notation, we suggest to introduce coherent notation as follows. Let $\chi_w^e(G)$ denote the minimum k such that there is a k -edge-weighting of G that vertex-colors G by the sums of weights of the edges incident to the vertex, $\chi_m^e(G)$ the minimum k such that there is a k -edge-weighting of G that vertex-colors G by multisets of weights (of the edges incident to the vertex) and finally $\chi_s^e(G)$ the minimum k such that there is a k -edge-weighting of G that vertex-colors G by sets of weights (of the edges incident to the vertex). (The last one was denoted by $\text{gndi}(G)$ in [5], and called the general neighbor distinguishing index of G , but we think it should be fit into this more consistent terminology and notation.)

Karoński et al. [7] initiated the study of both $\chi_m^e(G)$ and $\chi_w^e(G)$ and conjectured that both parameters are equal to 3, i.e., for every graph G without an edge component G has a 3-edge-weighting that is vertex-coloring by sums. Recently, Addario-Berry et al. [2] showed that for every graph G without an edge component, $\chi_m^e(G) \leq 4$, i.e., it has a 4-edge-weighting that is vertex-coloring by multisets and if the minimum degree is at least 1000 then $\chi_m^e(G) \leq 3$. Furthermore, Addario-Berry et al. [3] showed that for every graph G without an edge component, $\chi_w^e(G) \leq 16$, i.e., it has a 16-edge-weighting that is vertex-coloring by sums.

“Proper” edge-weightings have also been studied. That is, edge-weightings where no two incident edges get the same weight (i.e. edge-colorings) and

for any edge xy the set of edge-weights on edges incident to x is different from the set of edge-weights on edges incident to y . This graph parameter, the *neighbor-distinguishing index* or $\text{ndi}(G)$ in notation, was introduced by Zhang et al. in [8]. It is easy to see that $\text{ndi}(C_5) = 5$ and in [8] it is conjectured that $\text{ndi}(G) \leq \Delta(G) + 2$ for any connected graph $G \notin \{K_2, C_5\}$.

The conjecture has been confirmed by Balister et al. [4] for bipartite graphs and for graphs G with $\Delta(G) = 3$. The authors also showed that for every graph G without an edge component, $\text{ndi}(G) \leq \Delta(G) + O(\log \chi(G))$. Hatami [6] showed that for every graph G without an edge component and $\Delta(G) > 10^{20}$, then $\text{ndi}(G) \leq \Delta(G) + 300$.

The problem of determining $\chi_s^e(G)$ is deeper than it first seems to be: it is easy to see that to decide if $\chi_s^e(G) = 2$ for a bipartite graph G is equivalent to decide if the hypergraph formed by the neighborhoods of the vertices in one class of the bipartition is 2-colorable (has the **B**-property in other words). In [5], Gyóri et al. prove that if G is graph without an edge component, then

$$\chi_s^e(G) \leq 2\lceil \log_2 \chi(G) \rceil + 1.$$

The following theorem, also proved in [5], will assist us later in the proof of the main result. For an edge-weighting φ and a set X of vertices, let $S_\varphi(X)$ denote the family of all weight sets on vertices of X , i.e., $S_\varphi(X) = \{S_\varphi(x) \mid x \in X\}$.

Theorem 1. *If G is a bipartite graph without an edge component, then $\chi_s^e(G) \leq 3$. Furthermore, there is an edge-weighting φ and bipartite classes X, Y of G such that,*

$$\begin{aligned} S_\varphi(X) &\subseteq \{\{3\}, \{1, 2\}\}, \\ S_\varphi(Y) &\subseteq \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}. \end{aligned}$$

□

The main result of this paper is as follows.

Theorem 2. *If G is graph without an edge component and $\chi(G) \geq 3$, then*

$$\chi_s^e(G) = \lceil \log_2 \chi(G) \rceil + 1.$$

In [5] it is asked whether there exists a planar graph G with $\chi_s^e(G) > 3$. The Four Color Theorem combined with Theorem 1 and Theorem 2 imply that there is no such graph.

Corollary 3. *If G is a planar graph without an edge component, then*

$$\chi_s^e(G) \leq 3.$$

□

The proof of Theorem 2 will be separated into three parts. First we prove it for $\chi(G) \leq 4$, then for $5 \leq \chi(G) \leq 8$, and finally for $\chi(G) \geq 8$. The next section will be concerned with the proof of the upper bound on $\chi_s^e(G)$. The lower bound is a simple observation and will be used implicitly in the proofs in Section 2.

Remark 4. *If G is a graph without an edge component, then*

$$\chi_s^e(G) \geq \lceil \log_2 \chi(G) \rceil + 1.$$

Proof. Assume that we have a vertex-coloring edge-weighting of G with $k = \chi_s^e(G)$ weights, and so we have at most 2^k different weight sets appearing in G . This naturally gives us a proper vertex-coloring of G with 2^k colors. However, it is clear that a vertex with weight set S and a vertex with weight set $\{1, 2, \dots, k\} - S$ cannot be neighbors as the weight sets of neighbors must have a nonempty intersection (the weight of the edge connecting neighbors is necessarily in the intersection of their weight sets). Therefore, we can color such vertices with the same color and thus at most 2^{k-1} different colors are needed to color G . So, $\chi(G) \leq 2^{k-1}$ yields $\lceil \log_2 \chi(G) \rceil \leq k - 1$. □

The parameter $\chi_s^e(G)$ is not a monotone graph parameter under the addition of edges. For example, the path on 4 vertices, P_4 , has $\chi_s^e(P_4) = 3$ but the cycle on 4 vertices, C_4 , has $\chi_s^e(C_4) = 2$. However, our results imply that for graphs of chromatic number at least 3 the parameter $\chi_s^e(G)$ is in fact monotone. It is somewhat surprising that monotonicity seems to be difficult to prove directly. Let us refer to the fact that the proof for 3-chromatic graphs is considerably more difficult than the proof for 4-chromatic graphs.

2 Proof of Theorem 2

Additional notation used in the proof is mostly standard. In particular, the neighbors of a vertex v are denoted by $N(v)$. Similarly, the set of all neighbors of a set of vertices X is denoted by $N(X) = \bigcup_{v \in X} N(v) - X$. The set of all

edges between two disjoint sets of vertices X and Y is denoted by $E(X, Y)$. We will call an edge-weighting of G *canonical* if there is a proper coloring of the vertices with $\chi(G)$ colors such that the family of weight sets appearing on vertices in any color class is strictly disjoint from the family of weight sets appearing on vertices in another color class. Note that a canonical edge-weighting is necessarily vertex-coloring (but a vertex-coloring edge-weighting need not be canonical).

Theorem 2 will follow from three lemmas.

Lemma 5. *If G is graph without an edge component and $3 \leq \chi(G) \leq 4$, then*

$$\chi_s^e(G) = 3.$$

Lemma 6. *If G is a graph without an edge component and $5 \leq \chi(G) \leq 8$, then*

$$\chi_s^e(G) = 4.$$

Lemma 7. *If G is a graph without an edge component and $\chi(G) \geq 8$, then*

$$\chi_s^e(G) = \lceil \log_2 \chi(G) \rceil + 1.$$

2.1 Proof of Lemma 5

Let G be a 3-chromatic graph. We call a 3-coloring X, Y, Z of $V(G)$ *stable* if the following conditions hold:

- (i) if $x \in X$, then $N(x) \cap Y \neq \emptyset$
- (ii) if $y \in Y$, then $N(y) \cap Z \neq \emptyset$
- (iii) if $z \in Z$, then $N(z) \cap X \neq \emptyset$

In other words, each vertex in X must have a neighbor in Y , each vertex in Y must have a neighbor in Z and each vertex in Z must have a neighbor in X . If a 3-coloring is not stable, then call it *unstable* and observe that there must be a vertex that fails to satisfy the above requirement. We will call such a vertex *unstable*, otherwise a vertex is *stable*.

Proposition 8. *If G is 3-chromatic and has a stable 3-coloring X, Y, Z , then $\chi_s^e(G) = 3$. Furthermore, there is a canonical edge-weighting φ such that,*

$$\begin{aligned} S_\varphi(X) &\subseteq \{\{3\}, \{1, 3\}\}, \\ S_\varphi(Y) &\subseteq \{\{2\}, \{2, 3\}\}, \\ S_\varphi(Z) &\subseteq \{\{1\}, \{1, 2\}\}. \end{aligned}$$

Proof. If X, Y, Z is a stable 3-coloring of G then weight all $E(X, Y)$ edges with 3, all $E(Y, Z)$ edges with 2 and all $E(X, Z)$ edges with 1. Notice that among vertices of X the only possible weight sets are $\{3\}$ and $\{1, 3\}$, among vertices of Y the only possible weight sets are $\{2\}$ and $\{2, 3\}$ and among vertices of Z the only possible weight sets are $\{1\}$ and $\{1, 2\}$. Clearly this edge-weighting yields φ with the desired properties. \square

Therefore to show a 3-chromatic graph, G , has $\chi_s^e(G) = 3$ it is sufficient to find a stable 3-coloring.

Proposition 9. *If G is a connected 3-chromatic graph and contains a triangle, then G has a stable 3-coloring.*

Proof. Let X, Y, Z be the color classes of a 3-coloring of G . Orient the edges of G from X to Y , from Y to Z and from Z to X . Define $D \subset V(G)$ to be the set of vertices v for which either v is on a directed cycle in G or v is on a directed path to a directed cycle in G . Note that every vertex in D has outdegree at least 1 and therefore is stable.

Now choose a 3-coloring X, Y, Z that maximizes the size of D . If $D = V(G)$, then all vertices of G are stable and we are done. So, let us assume $|D| < |V(G)|$. Now let us recolor the vertices of G as follows: if $v \in D \cap X$ then put v in X' , if $v \in D \cap Y$ then put v in Y' , if $v \in D \cap Z$ then put v in Z' ; if $v \in X - D$ put v in Y' , if $v \in Y - D$ then put v in Z' , if $v \in Z - D$ then put v in X' . In other words, vertices in D keep their color and vertices of $V(G) - D$ are moved to the “next” color class. Now, orient the edges in this recoloring from X' to Y' , from Y' to Z' and from Z' to X' .

Let us confirm that this new coloring is proper. For the sake of contradiction assume it is not proper. For the coloring to be not proper there must be some edge contained in one of the color classes. Edges with both endpoints in D remain between different color classes as both endpoints stay in different color classes. Edges with both endpoints in $V(G) - D$ remain between different color classes as both endpoints move to their respective “next” classes. Therefore, we must examine edges of the form uw where $u \in D$ and $w \in V(G) - D$. Without loss of generality, let us assume that after recoloring the vertices of $V(G) - D$ that u and w are both in color class X' . This means that w moved from Z to X' and that before recoloring, the edge uw was oriented from w to u . But this means that w should have been in D not $V(G) - D$. This is a contradiction, therefore the recoloring is a proper coloring.

Now let us examine the maximality of D . After recoloring, let us define D' as the set of vertices on a directed cycle or on a directed path to a directed cycle under the new coloring X', Y', Z' . Note that all vertices in D are in D' as no vertices of D are recolored and therefore maintain their original orientation. Because G is connected and D is nonempty (it contains at least a triangle), there was a directed edge \overrightarrow{vu} such that $v \in D$ and $u \in V(G) - D$ under the original orientation. Without loss of generality, assume that $v \in X$. Thus, $u \in Y$ to force the appropriate orientation of vu . After recoloring, v is in X' and u is in Z' . Therefore, the edge vu is now oriented from u to v which implies that $u \in D'$. But $u \notin D$, so $|D'| > |D|$ contradicting the maximality of D . Therefore, we must have that $D = V(G)$ and we have a stable 3-coloring X, Y, Z . \square

We note that this proof is essentially the same as the proof of Lemma 2.1 in [2]. We also note that in Proposition 9, a triangle as an induced subgraph was important only in that it is a stable graph. With appropriate adjustments to the proof, the proposition can be restated with any stable induced subgraph instead of a triangle. However, as stated the proposition is strong enough for the remaining proofs.

A number of stable and unstable graphs are known. In particular, the Petersen Graph and odd cycles of length divisible by 3 are stable, while odd cycles of length not divisible by 3 are unstable.

We introduce the notion of an almost-canonical edge-weighting of a 3-chromatic graph to assist us in the proof of Theorem 2. We call an edge-weighting, φ , of a 3-chromatic graph *almost-canonical* if there exists a 3-coloring X, Y, Z and a vertex $x \in X$ such that the families of weight sets $S_\varphi(X - x), S_\varphi(Y), S_\varphi(Z)$ are pairwise disjoint and $S_\varphi(x) \not\subseteq S_\varphi(X)$. Note that an almost-canonical edge-weighting need not be vertex-coloring.

Proposition 10. *If G is an unstable connected 3-chromatic graph and x is an arbitrary vertex of G , then G has an almost-canonical 3-edge-weighting φ and a 3-coloring X, Y, Z with $x \in X$ such that,*

$$\begin{aligned} S_\varphi(X - x) &\subseteq \{\{3\}, \{1, 3\}\}, \\ S_\varphi(Y) &\subseteq \{\{2\}, \{2, 3\}\}, \\ S_\varphi(Z) &\subseteq \{\{1\}, \{1, 2\}\}, \\ S_\varphi(x) &= \{1\}. \end{aligned}$$

Proof. Let x be an arbitrary vertex of G . Let us add two new vertices v and w and three new edges xv , xw , and vw to G thus creating a triangle in this new graph. By Proposition 9, this new graph has a stable 3-coloring. Let X, Y, Z be the color classes of this coloring after removing v and w (and their incident edges) and assume without loss of generality that $x \in X$. Because we only removed neighbors of x from the stable graph, x is the only unstable vertex in the coloring of G , i.e., x has no neighbors in Y .

Now, weight all edges in $E(X, Y)$ with weight 3, all edges in $E(Y, Z)$ with weight 2 and all edges in $E(X, Z)$ with weight 1. The only possible weight sets in $X - x$ are $\{3\}$ and $\{1, 3\}$, the only possible weight sets in Y are $\{2\}$ and $\{2, 3\}$ and the only possible weight sets in Z are $\{1\}$ and $\{1, 2\}$ while x must have weight set $\{1\}$. Clearly this edge-weighting is almost-canonical and yields φ with the desired properties. \square

Proposition 11. *If G is an unstable connected 3-chromatic graph with a vertex of degree 1, then $\chi_s^e(G) = 3$.*

Proof. Let x be a vertex in of degree 1 in G . Call z the single neighbor of x . The vertex z has degree at least 2, otherwise xz is an isolated edge. Let us add two new vertices v and w and three new edges xv , xw , and vw to G thus creating a triangle in this new graph. By Proposition 9, this new graph has a stable 3-coloring. Let X, Y, Z be the color classes of this coloring after removing v and w (and their incident edges) and assume without loss of generality that $x \in X$. Because we only removed neighbors of x from the stable graph, x is the only unstable vertex in the coloring of G , i.e., x has no neighbors in Y . Therefore $z \in Z$. If the neighbors of z are all in X then we can move x to Y thus making x stable and keeping z stable. This gives a stable 3-coloring of G , a contradiction. So, we may assume z has a neighbor in Y .

Now, weight all edges in $E(X, Y)$ with weight 3, all edges in $E(Y, Z)$ with weight 2 and all edges in $E(X, Z)$ with weight 1. The only possible weight sets in $X - x$ are $\{3\}$ and $\{1, 3\}$, the only possible weight sets in Y are $\{2\}$ and $\{2, 3\}$, and the only possible weight sets in Z are $\{1\}$ and $\{1, 2\}$ while x must have weight set $\{1\}$. However, the only neighbor of x is z which has weight set $\{1, 2\}$. Therefore, this 3-edge-weighting is vertex-coloring. \square

Proposition 12. *If G is a connected 3-chromatic graph, then $\chi_s^e(G) = 3$.*

Proof. By the previous propositions, we may assume that G is triangle-free, the minimum degree of G is at least 2 and G does not have a stable 3-coloring.

Let x be an arbitrary vertex of G . Let us add two new vertices v and w and three new edges xv , xw , and vw to G thus creating a triangle in this new graph. By Proposition 9, this new graph has a stable 3-coloring. Let X, Y, Z be the color classes of this coloring after removing v and w (and their incident edges) and assume without loss of generality that $x \in X$. Because we only removed neighbors of x from the stable graph, x is the only unstable vertex in the coloring of G , i.e., x has no neighbors in Y . Now we will construct a vertex-coloring edge-weighting of G . Let $L \subset N(x) \subset Z$ be the set neighbors of x that themselves have no neighbors in Y (note that each vertex in L must have at least one neighbor other than x by the minimum degree assumption). Let $M = N(L) - x \subset X$ be the neighbors of L (they are necessarily in X) excluding x .

Claim 13. *There exists a vertex-coloring 3-edge-weighting φ of G such that,*

$$\begin{aligned} S_\varphi(X - M - x) &\subseteq \{\{3\}, \{1, 3\}\}, \\ S_\varphi(M) &\subseteq \{\{3\}, \{1, 3\}, \{1, 2, 3\}\}, \\ S_\varphi(x) &= \{1\}, \\ S_\varphi(Z - L) &\subseteq \{\{1\}, \{1, 2\}\}, \\ S_\varphi(L) &\subseteq \{\{1, 2\}, \{1, 3\}\}, \\ S_\varphi(Y) &\subseteq \{\{2\}, \{2, 3\}\}. \end{aligned}$$

Clearly the above claim implies the proposition. We will construct a 3-edge-weighting with the following families of weight sets and then confirm that it is indeed vertex-coloring by checking that no vertices in X with weight set $\{1\}$ or $\{1, 3\}$ have neighbors in Z with the same weight set. We will weight the edges of G by the following steps taking care to note possible weight sets after each step (if they could have been changed).

1. Weight all edges incident to x with weight 1. This immediately gives $S_\varphi(x) = \{1\}$. Also at this point, vertices in L have weight set $\{1\}$; vertices in $Z - L$ have weight set \emptyset or $\{1\}$; all other weight sets are \emptyset .
2. Consider the induced bipartite subgraph $G[M \cup Z]$. For every $v_0 \in M$ let $v_0v_1v_2v_3 \dots v_r$ be a shortest path from v_0 to a vertex $v_r \in Z - L$ if such a path exists. Note that because $G[M \cup Z]$ is bipartite, r is odd. Weight the edges $v_0v_1, v_2v_3, v_4v_5, \dots, v_{r-1}v_r$ with weight 1 and weight the edges $v_1v_2, v_3v_4, \dots, v_{r-2}v_{r-1}$ with weight 2 for all such minimum-length paths. Note that weighting edges in this way will never force an

edge to get weight 1 and 2 at the same time as this would contradict minimality of the path lengths. Furthermore, any vertex in L on such a minimal path will necessarily have all of its incident edges weighted in this step. (The last two statements can be checked easily by the reader using contradictive arguments.) At this point, vertices in M have weight set \emptyset or $\{1\}$ or $\{1, 2\}$; vertices in L have weight set $\{1\}$ or $\{1, 2\}$; vertices in $Z - L$ have weight set \emptyset or $\{1\}$.

3. Weight all unweighted edges in $G[M \cup L]$ with weight 3. At this point, vertices in M have weight set $\{3\}$ or $\{1\}$ or $\{1, 3\}$ or $\{1, 2\}$ or $\{1, 2, 3\}$; vertices in L have weight set $\{1, 3\}$ or $\{1, 2\}$. After this step all edges incident to vertices of L have been weighted.
4. Weight all edges between X and Y with weight 3. Except for x every vertex of X has a neighbor in Y , so the vertices in M have weight set $\{3\}$ or $\{1, 3\}$ or $\{1, 2, 3\}$; vertices in $X - M - x$ have weight set $\{3\}$; vertices in Y have weight set \emptyset or $\{3\}$. After this step all edges incident to vertices of M have been weighted.
5. Weight all edges between $X - M - x$ and Z with weight 1 (all edges between M and Z are already weighted). At this point, vertices in $X - M - x$ have weight set $\{3\}$ or $\{1, 3\}$; all vertices in Z have a neighbor in X , so the vertices of $Z - L$ have weight set $\{1\}$. After this step all edges incident to vertices of X have been weighted.
6. Weight all edges between Z and Y with weight 2. All vertices in Y have a neighbor in Z , so the vertices of Y have weight set $\{2\}$ or $\{2, 3\}$; the vertices in $Z - L$ have weight set $\{1\}$ or $\{1, 2\}$. After this step all edges have been weighted.

At this point we have achieved the weight sets necessary for Claim 13. Now it remains to confirm that the edge-weighting given is vertex-coloring. In most cases this is immediate from the construction of the weighting. However, we must check that no vertex in X with weight set $\{1\}$ or $\{1, 3\}$ has a neighbor in Z with the same weight set. We distinguish two cases.

1. Weight set $\{1\}$. This weight set appears in X only on the vertex x . The neighbors of x are either in L or $Z - L$. The weight set $\{1\}$ does not appear in L . In $Z - L$ the weight set $\{1\}$ does appear, but only on vertices that have no neighbors in Y . If such a vertex were a neighbor

of x then it would have been in L initially. So, there are no edges in G with weight set $\{1\}$ on both endpoints.

2. Weight set $\{1, 3\}$. This weight set appears in M , $X - M - x$ and $L \subset Z$. However, M consists of the neighbors of L (excluding x) in X , so there are no edges between $X - M - x$ and L and we may restrict our analysis to edges between M and L . Let $l \in L$ and let $M_l \subset M$ be the neighbors of l in M . The only way for an edge with an endpoint l to get weight set $\{1, 3\}$ is to not be on a minimal path from M to $Z - L$. Otherwise, the weight set of l would necessarily contain a 2 as we alternate weights 1 and 2 along such minimal paths (see Step 2). This means that every vertex of M_l does not have a path to $Z - L$ and therefore every edge with an endpoint in M_l gets weight 3. So, the vertices of M_l all have weight set $\{3\}$. Therefore, there are no edges in G with weight set on $\{1, 3\}$ on both endpoints.

Therefore the given 3-edge-weighting is vertex-coloring, thus proving our claim and proposition. \square

We can use the existence of almost-canonical edge-weightings of 3-chromatic graphs to help show that for 4-chromatic graphs G we have $\chi_s^e(G) = 3$.

Proposition 14. *If G is a connected 4-chromatic graph, then $\chi_s^e(G) = 3$. Furthermore, there is a canonical edge-weighting φ and a 4-coloring X, Y, Z, W such that,*

$$\begin{aligned} S_\varphi(X) &\subseteq \{\{3\}, \{1, 3\}\}, \\ S_\varphi(Y) &\subseteq \{\{2\}, \{2, 3\}\}, \\ S_\varphi(Z) &\subseteq \{\{1\}, \{1, 2\}\}, \\ S_\varphi(W) &= \{\{1, 2, 3\}\}. \end{aligned}$$

Proof. Let us 4-color G such that a color class W is minimal over all 4-colorings. This implies that any vertex of W has a neighbor in each of the other three color classes (even if they are properly recolored). Consider the induced subgraph $G' = G[V(G) - W]$ on the other three color classes. Let us assume G' is connected (if it is not the next steps should be performed for each component). If G' is stable, then by Proposition 8 we have a stable

3-coloring X, Y, Z of G' and a canonical edge-weighting φ such that,

$$\begin{aligned} S_\varphi(X) &\subseteq \{\{3\}, \{1, 3\}\}, \\ S_\varphi(Y) &\subseteq \{\{2\}, \{2, 3\}\}, \\ S_\varphi(Z) &\subseteq \{\{1\}, \{1, 2\}\}. \end{aligned}$$

If G' is unstable, let x be a vertex of G with a neighbor in W . By Proposition 10, we have a 3-coloring X, Y, Z of G' and an almost-canonical edge-weighting φ such that,

$$\begin{aligned} S_\varphi(X - x) &\subseteq \{\{3\}, \{1, 3\}\}, \\ S_\varphi(Y) &\subseteq \{\{2\}, \{2, 3\}\}, \\ S_\varphi(Z) &\subseteq \{\{1\}, \{1, 2\}\}, \\ S_\varphi(x) &= \{1\}. \end{aligned}$$

In both cases, we extend the edge-weighting φ by weighting all $E(X, W)$ edges with 3, all $E(Y, W)$ edges with 2 and all $E(Z, W)$ edges with 1. This does not change the families on weight sets of vertices in $X - x, Y$ or Z . Furthermore, the weight set of x will necessarily become $\{1, 3\}$ as x has a neighbor in W . Because each vertex of W has a neighbor in X, Y, Z every vertex in W will necessarily have weight set $\{1, 2, 3\}$. This gives φ that satisfies the conditions of the proposition. \square

Clearly, Proposition 12 and Proposition 14 imply Lemma 5.

2.2 Proof of Lemma 6

Before proving Lemma 6, we introduce some definitions. Let X_1, X_2, \dots, X_t be pairwise disjoint independent sets of $V(G)$ (e.g. color classes in a coloring of G), let w_1, w_2, \dots, w_t be distinct edge weights and let $S_\varphi(X_i)$ be the family of weight sets appearing on vertices in X_i under a vertex-coloring edge-weighting φ of G . For $i = 1, 2, \dots, t$ we say that each X_i is w_i -safe if for any pair $a, b \in [1, t]$ where $a \neq b$ and any pair of weight sets $S_1 \in S_\varphi(X_a)$ and $S_2 \in S_\varphi(X_b)$ we have that $S_2 \neq S_1 \cup \{w_a\} \neq S_2 \cup \{w_b\} \neq S_1$. In particular, this implies that for all $i \in [1, t]$ we may add weight w_i to any weight sets in $S_\varphi(X_i)$ and not disrupt the vertex-coloring property. This tool will allow us to weight the edges of a graph in an inductive way without ruining previously “good” weightings.

Additionally, an independent set X is called i -free (with respect to φ) if $i \notin S$ for all $S \in S_\varphi(X)$, i.e., i never appears in the weight set of any vertex $x \in X$. The following proposition implies Lemma 6 and will form the base case of the inductive proof of Lemma 7.

Proposition 15. *If G is a graph such that $5 \leq \chi(G) \leq 8$, then $\chi_s^e(G) = 4$. Furthermore, if $\chi(G) = 8$ then there is a vertex-coloring edge-weighting of G and an 8-coloring of G with distinct color classes X_1, X_2, X_3, X_4, Y such that X_i is i -safe (for $i = 1, 2, 3, 4$) and Y is 4-free.*

Proof. We explicitly construct an edge-weighting of G where $\chi(G) = 8$. It will be clear from the proof that our weighting will also work for graphs of chromatic number between 5 and 8. Color G with 8 colors in such a way as to maximize the size of the subgraph H induced by the first four colors. Let $F = G[V(G) - V(H)]$ be the graph induced by the remaining color classes. Therefore, $|V(F)|$ is minimal over all colorings and no vertex of F can be colored with a color from H , i.e., each vertex in F has a neighbor in each color class of H .

The subgraph H is not necessarily be connected. We will distinguish 5 types of components of H . Let H_4 be an arbitrary 4-chromatic component of H , let H_3 be an arbitrary 3-chromatic component of H , let $H_2 \neq K_2$ be an arbitrary bipartite component of H , let xy be an arbitrary isolated edge of H and let v be an arbitrary isolated vertex of H . We will describe how to weight edges among these subgraphs. This technique should be followed for all such components.

By Proposition 14 we have a vertex-coloring edge-weighting, φ_4 , of H_4 with $\chi_s^e(H_4) = 3$ weights and a 4-coloring X_4, Y_4, Z_4, W_4 such that,

$$\begin{aligned} S_{\varphi_4}(X_4) &\subseteq \{\{3\}, \{1, 3\}\}, \\ S_{\varphi_4}(Y_4) &\subseteq \{\{2\}, \{2, 3\}\}, \\ S_{\varphi_4}(Z_4) &\subseteq \{\{1\}, \{1, 2\}\}, \\ S_{\varphi_4}(W_4) &= \{\{1, 2, 3\}\}. \end{aligned}$$

If H_3 has a stable 3-coloring then by Proposition 8 then we have a vertex-coloring edge-weighting, φ_3 , of H_3 with $\chi_s^e(H_3) = 3$ weights and a 3-coloring X_3, Y_3, Z_3 such that,

$$\begin{aligned} S_{\varphi_3}(X_3) &\subseteq \{\{3\}, \{1, 3\}\}, \\ S_{\varphi_3}(Y_3) &\subseteq \{\{2\}, \{2, 3\}\}, \\ S_{\varphi_3}(Z_3) &\subseteq \{\{1\}, \{1, 2\}\}. \end{aligned}$$

If H_3 does not have a stable 3-coloring then choose a vertex u in H_3 that has a neighbor in F (such a vertex exists as G is connected). Now, by Proposition 10 we have an almost-canonical edge-weighting, φ_3 , of H_3 with $\chi_s^e(H_3) = 3$ weights and a 3-coloring X_3, Y_3, Z_3 with $u \in X_3$ such that,

$$\begin{aligned} S_{\varphi_3}(X_3 - u) &\subseteq \{\{3\}, \{1, 3\}\}, \\ S_{\varphi_3}(Y_3) &\subseteq \{\{2\}, \{2, 3\}\}, \\ S_{\varphi_3}(Z_3) &\subseteq \{\{1\}, \{1, 2\}\}, \\ S_{\varphi_3}(u) &= \{1\}. \end{aligned}$$

Note that u may have the same weight set as some of its neighbors in Z_3 . (We will resolve this conflict later when weighting the edges from u to F .)

By Theorem 1 we have a vertex-coloring edge-weighting, φ_2 , of H_2 with $\chi_s^e(H_2) = 3$ weights and a bipartition X_2, Y_2 such that,

$$\begin{aligned} S_{\varphi_2}(X_2) &\subseteq \{\{3\}, \{1, 2\}\}, \\ S_{\varphi_2}(Y_2) &\subseteq \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}. \end{aligned}$$

Finally, we will weight all isolated edges xy with weight 2. We refer to this edge-weighting of all of the components of H (and hence H itself) as φ .

Now let:

$$\begin{aligned} X &= X_4 \cup X_3 \cup X_2 \cup \{x\} \cup \{v\}, \\ Y &= Y_4 \cup Y_3 \cup Y_2 \cup \{y\}, \\ Z &= Z_4 \cup Z_3, \\ W &= W_4. \end{aligned}$$

Note that X, Y, Z, W is a 4-coloring of H . By our choice of H , each vertex of F has a neighbor in each of X, Y, Z, W . Let A_1, A_2, A_3, A_4 be the color classes of F (if $\chi(G) < 8$ we just follow the given edge-weighting and ignore the steps involving the appropriate color classes A_i). Let us assume the color classes of F are colored such that each vertex in A_i has a neighbor in each A_j for all $j < i$. Let us weight all edges $E(A_4, A_3)$ with weight 2. Let us weight all other edges in F with weight 4.

Now it remains to weight all edges in $E(H, F)$. The table below describes how to weight edges between different color classes and shows what the possible weight sets are in each color class after weighting all of the edges of G . In particular, the first column represents each color class (some are split

into distinct components). Recall that each vertex in A_i has a neighbor in each of X, Y, Z, W . Columns two through five represent the weight to give edges between A_i and the corresponding color class in a given row (when such an edge exists). We refer to the weighting of all of the edges of G by ψ . The final column represents the possible weight sets appearing in the corresponding color class in a given row.

					S_ψ – possible weight sets
	A_1	A_2	A_3	A_4	
X_4, X_3	3	3	3	3	$\{3\}, \{1, 3\}$
X_2	3	3	3	3	$\{3\}, \{1, 2\}, \{1, 2, 3\}$
$\{x\}$	3	3	3	3	$\{2\}, \{2, 3\}$
$\{v\}$	3	3	3	3	$\{3\}$
Y_4, Y_3	3	3	3	3	$\{2\}, \{2, 3\}$
Y_2	3	3	3	3	$\{1\}, \{2\}, \{1, 3\}, \{2, 3\}$
$\{y\}$	4	4	4	4	$\{2\}, \{2, 4\}$
Z_4, Z_3	4	4	4	4	$\{1\}, \{1, 2\}, \{1, 4\}, \{1, 2, 4\}$
W_4	3	1	2	1	$\{1, 2, 3\}$
A_1		4	4	4	$\{3, 4\}$
A_2	4		4	4	$\{1, 3, 4\}$
A_3	4	4		2	$\{2, 3, 4\}$
A_4	4	4	2		$\{1, 2, 3, 4\}$

Now, by examining the table, we will verify that ψ is vertex-coloring. Clearly the single weight set appearing in A_i (with respect to ψ) only appears in A_i for each i . It remains to check that under the weighting ψ the weight sets on vertices in H are distinct from the weight sets on their neighbors in H . To show this we must confirm that there is no edge where both endpoints have the same weight set under ψ . Recall that each edge is between some pair of color classes X, Y, Z, W while being contained in one of the component types described above, i.e., H_4, H_3, H_2, xy . With this and the contents of the last column of the table, the only case where an edge might get the same weight set on both of its endpoints is when it is of the form $K_2 = xy$ in H . For such a component, we have $S_\psi(x) \in \{\{2\}, \{2, 4\}\}$ and $S_\psi(y) \in \{\{2\}, \{2, 3\}\}$ but both x and y cannot get weight set $\{2\}$ as G has no K_2 components, i.e., at least one of x or y has a neighbor in F (and all edges from F to x get weight 3 and all edges from F to y get weight 4). In no other case is there

danger of an edge having the same weight set on both endpoints. Therefore, our edge-weighting is vertex-coloring.

Furthermore, it is clear that when $\chi(G) = 8$ we have that A_2 is 1-safe, A_3 is 2-safe, A_4 is 3-safe, A_1 is 4-safe and W is 4-free (all with respect to the final weighting ψ) thus giving the proposition. \square

2.3 Proof of Lemma 7

For technical reasons we prove the following stronger version of Lemma 7.

Proposition 16. *Suppose G is a graph such that $\chi(G) \geq 8$ and let $k = \lceil \log_2 \chi(G) \rceil$, then $\chi_s^e(G) \leq k + 1$. Furthermore, there exists a vertex-coloring $(k + 1)$ -edge-weighting of G and a $\chi(G)$ -coloring of G with distinct color classes $X_1, X_2, \dots, X_{k+1}, Y$ such that X_i is i -safe (for $i = 1, 2, \dots, k + 1$) and Y is $(k + 1)$ -free.*

Proof. Let G be a graph with chromatic number $\chi(G)$. There exists an integer k such that $2^{k-1} < \chi(G) \leq 2^k$. We proceed by induction on $\chi(G)$. The base case $\chi(G) = 8$ holds by Proposition 15. So, let $\chi(G) > 8$ and assume the statement of the proposition for all graphs H with $\chi(H) < \chi(G)$.

Color G with $\chi(G)$ colors in such a way as to maximize the size of the subgraph H induced by the first 2^{k-1} colors. Let $F = G[V(G) - V(H)]$ be the graph induced by the remaining color classes. Therefore, $|V(F)|$ is minimal over all colorings and no vertex of F can be colored with a color from H , i.e., every vertex in F has a neighbor in each color class of H . By induction we have $\chi_s^e(H) = k$ and we have an k -edge-weighting of H and a (2^{k-1}) -coloring of H with distinct color classes X_1, X_2, \dots, X_k, Y such that X_i is i -safe (for $i = 1, 2, \dots, k$) and Y is k -free. Let us keep this edge-weighting of $H \subset G$ and weight the remaining edges of G .

First, weight all edges in F with (new) weight $k + 1$. Now it remains to weight the edges between H and F . Label the color classes of F with $(k - 1)$ -length binary strings from 0 to $\chi(G) - 2^{k-1}$. Let $v \in F$ be an arbitrary vertex in F . By construction of H and F , v has a neighbor in each color class of H (notably in each X_i and Y). If the binary string corresponding to the color class of v has a 1 in the i -th binary digit (i ranges from 1 to $k - 1$) then weight all edges between v and X_i with weight i . Next, weight all edges between v and X_k with k and all edges between v and Y with $k + 1$ (this guarantees that each weight set in F has both weights k and $k + 1$). Finally, for all remaining unweighted edges $vw \in E(F, H)$ we weight vw as follows:

if $w \in H$ is incident to an edge with weight k then weight vw with weight k . Otherwise, weight vw with weight $k+1$. In this way, we guarantee that every weight set in H has at most one of the weights k and $k+1$. This immediately distinguishes the weight sets in H from those in F . Clearly, each color class in F will have a single unique weight set corresponding to its (unique) binary string (and the weights k and $k+1$). The color classes of H were already distinguished by the first k weights. The edges between F and H only added weight i to i -safe color classes of H (for $1 \leq i \leq k$) or a new weight $k+1$, so weight sets of any pair adjacent vertices in H remain distinct. This gives a vertex-coloring $(k+1)$ -edge-weighting of G where $k+1 = \lceil \log_2 \chi(G) \rceil + 1$. Furthermore, for $i = 1, 2, \dots, k-1$ the color class X_i remains i -safe, the first class of F (its corresponding binary string is $00 \dots 0$) is k -safe, class Y is now $(k+1)$ -safe class and X_k is $(k+1)$ -free (as all edges between X_k and F got weight k). \square

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References

- [1] L. Addario-Berry, K. Dalal, C. McDiarmid, B. Reed and A. Thomason, Vertex-colouring edge-weightings, *Combinatorica* **27** (2007), 1-12.
- [2] L. Addario-Berry, R.E.L. Aldred, K. Dalal, and B. Reed, Vertex colouring edge partitions, *Journal of Combinatorial Theory Ser. B*, 94 (2005), 237-244.
- [3] L. Addario-Berry, K. Dalal, B. A. Reed, Degree constrained subgraphs, *Electron. Notes Discrete Math.* **19** (2005), 257-263.
- [4] P.N. Balister, E. Győri, J. Lehel, and R.H. Schelp, Adjacent vertex distinguishing edge-colorings, *SIAM J. on Discrete Math.* **21** (2007), 237-250.

- [5] E. Győri, M. Horňák, C. Palmer and M. Woźniak, General neighbour-distinguishing index of a graph, to appear in *Discrete Math.*
- [6] H. Hatami, $\Delta + 300$ is a bound on the adjacent vertex distinguishing edge chromatic number, *J. Combin. Theory Ser. B* **95** (2005), 246-256.
- [7] M. Karoński, T. Łuczak and A. Thomason, Edge weights and vertex colours, *J. Combinatorial Theory B* 91 (2004) 151-157.
- [8] Z. Zhang, L. Liu and J. Wang, Adjacent strong edge coloring of graph, *Appl. Math. Lett.* **15** (2002), 623-626.