General neighbour-distinguishing index of a graph

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Abstract

It is proved that edges of a graph G with no component K_2 can be coloured using at most $2\lceil \log_2 \chi(G) \rceil + 1$ colours so that any two adjacent vertices have distinct sets of colours of their incident edges.

Keywords: Edge colouring; Colour set; General neighbour-distinguishing index

1 Introduction

All graphs we deal with in this paper are simple and finite. Let G be a graph and k a non-negative integer. A (*general*) *k-edge-colouring* of G is a mapping $\varphi : E(G) \to \bigcup_{i=1}^{k} \{i\}$. The *colour set* (with respect to φ) of a vertex $x \in V(G)$ is the set $S_\varphi(x)$ of colours (values of φ) of edges incident to x. The colouring

*Partially supported by the Hungarian OTKA Grant T 34702.

†This work was supported by Science and Technology Assistance Agency under the contract No. APVT-20-004104.

φ is *neighbour-distinguishing* if $S_\varphi(x) \neq S_\varphi(y)$ whenever vertices x, y are adjacent. A neighbour-distinguishing colouring will be frequently shortened to an *nd-colouring*. The *general neighbour-distinguishing index* of G is the minimum k in a general k-edge-colouring of G that is neighbour-distinguishing, and will be denoted by $\mathrm{gndi}(G)$. If G has a component K_2, then G does not have any nd-colouring, hence for the sake of the completeness of the definition in such a case we set $\mathrm{gndi}(G) := \infty$. Having in mind the following evident statement, the analysis of the general neighbour-distinguishing index can be restricted to connected graphs.

Proposition 1. *If $n \geq 2$ and G is a disconnected graph with components $G_1, \dots, G_n$, then* $\mathrm{gndi}(G) = \max\ (\mathrm{gndi}(G_i) : i = 1, \dots, n)$. □

The general neighbour-distinguishing index is a relaxation of two known graph invariants. If $S_\varphi(x) \neq S_\varphi(y)$ is required for any two distinct vertices x, y, the corresponding parameter $\chi_0(G)$, called the *point-distinguishing chromatic index* of G, has been introduced by Harary and Plantholt in [3]. The authors proved, among other things, that $\chi_0(K_n) = \lceil \log_2 n \rceil + 1$ for any $n \geq 3$. In spite of the fact that the structure of complete bipartite graphs is simple, it seems that the problem of determining $\chi_0(K_{m,n})$ is not easy, especially in the case $m = n$, as documented by papers of Zagaglia Salvi [8], [9], Horňák and Soták [5], [6] and Horňák and Zagaglia Salvi [7].

On the other hand, if only *proper* nd-colourings are considered, the *neighbour-distinguishing index* of G, in notation $\mathrm{ndi}(G)$, is obtained. This invariant has been introduced only recently by Zhang et al. in [10]. It is easy to see that $\mathrm{ndi}(C_5) = 5$ and in [10] it is conjectured that $\mathrm{ndi}(G) \leq \Delta(G) + 2$ for any connected graph $G \notin \{K_2, C_5\}$. The conjecture has been confirmed by Balister et al. in [1] for bipartite graphs and for graphs G with $\Delta(G) = 3$. Edwards et al. in [2] have shown even that $\mathrm{ndi}(G) \leq \Delta(G) + 1$ if G is bipartite, planar, and of maximum degree $\Delta(G) \geq 12$. In the general case a weaker statement $\mathrm{ndi}(G) \leq \Delta(G) + 300$ has been proved by Hatami in [4] for all graphs G with $\Delta(G) > 10^{20}$.

For integers p, q we denote by $[p, q]$ the *integer interval* lower bounded by p and upper bounded by q, i.e., $[p, q] := \bigcup_{i=p}^{q} \{i\}$. Let n and $l_1, \dots, l_n$ be non-negative integers. The *concatenation* of finite sequences $A_i = (a_i^1, \dots, a_i^{l_i})$, $i = 1, \dots, n$, is defined as the sequence $\prod_{i=1}^{n} A_i := (a_1^1, \dots, a_1^{l_1}, \dots, a_n^1, \dots, a_n^{l_n})$. If $A_i = A$ for each $i \in [1, n]$, we write A^n instead of $\prod_{i=1}^{n} A$. If $n = 0$, A^n is the empty sequence ( ).

Let G be a graph let $x, y \in V(G)$. By $\deg_G(x)$ we denote the degree of x in G and by $d_G(x, y)$ the distance between x and y in G. An *arm* of a tree T is a maximal (non-extendable) subpath A of T such that $\deg_A(x) = \deg_T(x) = 2$ for any internal vertex $x \in V(A)$ and $V(A)$ contains a pendant vertex of T. Let $a(T)$ denote the number of arms of T. If T is (isomorphic to) an n-vertex path P_n, then $a(T) = 1$ and T itself is the only arm of T. On the other hand, if $\Delta(T) \geq 3$, any arm A of T has one endvertex of degree one, the other of degree at least three and $a(T)$ is equal to the number of pendant vertices of T.

The main goal of this paper is to show that $\mathrm{gndi}(G) \leq 2\lceil \log_2 \chi(G) \rceil + 1$ for any graph G having no isolated edges.

2 Paths, cycles and bipartite graphs

Proposition 2. *For any graph G the following statements are equivalent:*

(1) $\mathrm{gndi}(G) = 2$.

(2) *G is bipartite and there is a bipartition $\{X_1 \cup X_2, Y\}$ of $V(G)$ such that $X_1 \cap X_2 = \emptyset$ and any vertex of Y has at least one neighbour in both X_1 and X_2.*

Proof. (1) $\Rightarrow$ (2): Consider an nd-colouring $\varphi : E(G) \to [1, 2]$. The only three available non-empty colour sets are $\{1\}$, $\{2\}$ and $\{1, 2\}$. Since $\{1\} \cap \{2\} = \emptyset$, for any $xy \in E(G)$ exactly one of $S_\varphi(x)$ and $S_\varphi(y)$ is equal to $\{1, 2\}$. Let $Y := \{y \in V(G) : S_\varphi(y) = \{1, 2\}\}$ and let $X_i := \{x \in V(G) : S_\varphi(x) = \{i\}\}$, $i = 1, 2$. Clearly, $X_1 \cap X_2 = \emptyset$, $(X_1 \cup X_2) \cap Y = \emptyset$, any edge of G joins a vertex of $X_1 \cup X_2$ to a vertex of Y, and any vertex of Y has at least one neighbour in both X_1 and X_2.

(2) $\Rightarrow$ (1): Let the colouring $\varphi : E(G) \to [1, 2]$ be defined so that $\varphi(xy) = i$ if and only if $x \in X_i$ and $y \in Y$, $i = 1, 2$. Then $S_\varphi(x) = \{i\}$ for any $x \in X_i$, $i = 1, 2$, $S_\varphi(y) = \{1, 2\}$ for any $y \in Y$, and so φ is neighbour-distinguishing. □

An nd-colouring $\varphi : E(G) \to [1, 3]$ of a bipartite graph G is said to be *canonical* if there is a *canonical ordered bipartition* (X, Y) of $V(G)$, one that satisfies $S_\varphi(x) \in \mathcal{S}_1 := \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}$ for every $x \in X$ and $S_\varphi(y) \in \mathcal{S}_2 := \{\{3\}, \{1, 2\}\}$ for every $y \in Y$. The set $\mathcal{S}_1$ has the following important property: whenever $S \in \mathcal{S}_1$, then also $S \cup \{3\} \in \mathcal{S}_1$. A canonical nd-colouring φ of a tree T is 3-*canonical* if $S_\varphi(v) \neq \{3\}$ for any vertex

$v \in V(T)$ with $\deg_T(v) \geq 2$. A 3-canonical nd-colouring φ of a path P_n is $(3,i)$-*canonical*, $i \in [1,2]$, if there is a pendant edge $e \in E(P_n)$ such that $\varphi(e) = i$.

Proposition 3. *Let n be an integer, $n \geq 3$, and let $i \in [1,2]$.*

(1) *If n is odd, then* $\text{gndi}(P_n) = 2$ *and there is a $(3,i)$-canonical* nd-*colouring $\varphi : E(P_n) \to [1,2]$.*

(2) *If n is even, then* $\text{gndi}(P_n) = 3$ *and there is a $(3,i)$-canonical* nd-*colouring $\varphi : E(P_n) \to [1,3]$.*

Proof. Suppose that $\text{gndi}(P_n) = 2$ and let $\{X_1 \cup X_2, Y\}$ be the bipartition of $V(P_n)$ yielded by Proposition 2. The natural sequence of vertices of P_n (from one endvertex to the other) is an alternating sequence of vertices from $X_1 \cup X_2$ and Y that starts and ends with a vertex of $X_1 \cup X_2$. Therefore $|X_1 \cup X_2| = |Y| + 1$ and n is odd.

(1) If $n = 4j - 1$, then the nd-colouring, determined by the sequence $(1,2)(2,1,1,2)^{j-1}$ of colours of consecutive edges of P_n, is both $(3,1)$-canonical and $(3,2)$-canonical. If $n = 4j + 1$, the nd-colouring, corresponding to $(1,2,2,1)^j$, is $(3,1)$-canonical.

(2) If $n = 4j$ or $n = 4j + 2$, then the sequence $(3,2,1)(1,2,2,1)^{j-1}$ or $(3)(1,2,2,1)^j$, respectively, represents a $(3,1)$-canonical nd-colouring of P_n.

If φ is a $(3,1)$-canonical nd-colouring of P_n, then the colouring $\tilde{\varphi}$, defined by $\varphi(e) = 3 \Rightarrow \tilde{\varphi}(e) = 3$ and $\varphi(e) = k \in [1,2] \Rightarrow \tilde{\varphi}(e) = 3 - k$, is a $(3,2)$-canonical nd-colouring of P_n, and uses the same number of colours as φ does. □

Proposition 4. *Let n be an integer, $n \geq 3$.*

(1) *If $n \equiv 0 \pmod 4$, then* $\text{gndi}(C_n) = 2$.

(2) *If $n \not\equiv 0 \pmod 4$, then* $\text{gndi}(C_n) = 3$.

Proof. Suppose that $\text{gndi}(C_n) = 2$ and let $\{X_1 \cup X_2, Y\}$ be the bipartition of $V(C_n)$ from Proposition 2. Pick a vertex $y \in Y$, take his unique neighbour $x_1 \in X_1$ and consider the natural sequence of vertices of C_n given by the ordered pair (y, x_1) that ends with the other neighbour $x_2 \in X_2$ of y. This sequence is built up by concatenating ordered 4-tuples of vertices belonging successively to Y, X_1, Y and X_2, hence $n \equiv 0 \pmod 4$.

The following (cyclic) sequences represent an nd-colouring of C_n with minimum possible number of colours successively for $n = 4j-1$, $4j$, $4j+1$ and $4j+2$: $(1,2,3)(1,2,2,1)^{j-1}$, $(1,2,2,1)^j$, $(1,2,2,3,1)(1,2,2,1)^{j-1}$, $(1,2,3)^2(1,2,2,1)^{j-1}$. □

Theorem 5. *If T is a tree with $|E(T)| \geq 2$, then* $\mathrm{gndi}(T) \leq 3$ *and there is a 3-canonical nd-colouring of T.*

Proof. We proceed by induction on $a(T)$. If $a(T) = 1$, there is $n \geq 3$ such that $T \simeq P_n$ and we are done by Proposition 3.

Suppose that $a(T) > 1$ and there is a 3-canonical nd-colouring of an arbitrary tree T' with $a(T') < a(T)$. Consider a pendant vertex $x \in V(T)$ and such a vertex $y \in V(T)$ with $\deg_T(y) \geq 3$ that $d_T(x, y)$ is minimal. The subpath A of T with endvertices x and y is an arm of T and $T' := T - (V(A) - \{y\})$ is a subtree of T with $a(T') = a(T) - 1$ and $|E(T')| \geq 2$. By the induction hypothesis there is a 3-canonical nd-colouring $\varphi' : E(T') \to [1, 3]$. Let (X', Y') be a canonical ordered bipartition of $V(T')$ (there is one corresponding to φ'). A 3-canonical nd-colouring $\psi : E(T) \to [1, 3]$ will be found as a continuation of φ'.

(1) $V(A) = \{x, y\}$

(11) If $S_{\varphi'}(y) \neq \{1, 2\}$, then $S_{\varphi'}(y) \in \mathcal{S}_1$. Defining $\psi(xy) := 3$ yields $S_\psi(y) = S_{\varphi'}(y) \cup \{3\} \in \mathcal{S}_1$, $S_\psi(x) = \{3\} \in \mathcal{S}_2$ and $(X', Y' \cup \{x\})$ is the canonical ordered bipartition of $V(T)$.

(12) If $S_{\varphi'}(y) = \{1, 2\}$, set $\psi(xy) := 1$. Then $S_\psi(x) = \{1\} \in \mathcal{S}_1$, $S_\psi(y) = \{1, 2\} \in \mathcal{S}_2$ and $(X' \cup \{x\}, Y')$ is the canonical ordered bipartition of $V(T)$.

(2) Provided that $|V(A)| \geq 3$, let z be the unique neighbour of y in A. Since $\deg_{T'}(y) = \deg_T(y) - 1 \geq 2$ and the colouring φ' is 3-canonical, there is $i \in S_{\varphi'}(y) \cap [1, 2]$. By Proposition 3 there exists a $(3, i)$-canonical nd-colouring $\varphi : E(A) \to [1, 3]$ with $\varphi(yz) = i$. Clearly, if (X, Y) is the canonical ordered bipartition of $V(A)$, then $y \in X$, $z \in Y$ and $S_\varphi(z) = \{1, 2\}$.

(21) If $S_{\varphi'}(y) \neq \{1, 2\}$, let ψ be the common continuation of both φ' and φ. In such a case $S_\psi(v) = S_{\varphi'}(v)$ for any $v \in V(T')$, $S_\psi(v) = S_\varphi(v)$ for any $v \in V(A) - \{y\}$ and the canonical ordered bipartition of $V(T)$ is $(X' \cup X, Y' \cup Y)$.

(22) If $S_{\varphi'}(y) = \{1, 2\}$, then $y \in Y'$.

(221) If $V(A) = \{x, y, z\}$, set $\psi(yz) := 2$ and $\psi(zx) := 3$ to obtain $S_\varphi(y) = \{1, 2\} \in \mathcal{S}_2$, $S_\psi(z) = \{2, 3\} \in \mathcal{S}_1$ and $S_\psi(x) = \{3\} \in \mathcal{S}_2$; the canonical ordered bipartition of $V(T)$ is $(X' \cup \{z\}, Y' \cup \{x\})$.

(222) If $|V(A)| \geq 4$, then $A^- := A - y$ is a path on $|V(A)| - 1 \geq 3$ vertices. By Proposition 3 there is a $(3, 1)$-canonical nd-colouring $\varphi^- : E(A^-) \to [1, 3]$ such that $S_{\varphi^-}(z) = \{1\}$; if (X^-, Y^-) is the canonical ordered bipartition of $V(A^-)$, then $z \in X^-$. The continuation ψ of both φ' and φ^- with $\psi(yz) := 1$ satisfies $S_\psi(v) = S_{\varphi'}(v)$ for any $v \in V(T')$, $S_\psi(v) = S_{\varphi^-}(v)$ for any $v \in V(A^-)$

and $(X' \cup X^-, Y' \cup Y^-)$ is the canonical ordered bipartition of $V(T)$. $\square$

Theorem 6. *If G is a connected bipartite graph with $|E(G)| \geq 2$, then* $\text{gndi}(G) \leq 3$ *and G has a canonical nd-colouring.*

Proof. We prove the theorem by induction on the cyclomatic number $\mu(G) := |E(G)|-|V(G)|+1$. If $\mu(G) = 0$, then G is a tree and we can use Theorem 5. Assume that $\mu(G) > 0$ and there is a canonical nd-colouring of any connected bipartite graph H satisfying $|E(H)| \geq 2$ and $\mu(H) < \mu(G)$. From $\mu(G) > 0$ it follows that there is a cycle C in G (of an even length). If $xy \in E(C)$, then by the induction hypothesis for the connected graph $H := G - xy$ with $|E(H)| = |E(G)| - 1 \geq 3$ and $\mu(H) = \mu(G) - 1$ there exists a canonical nd-colouring $\varphi : E(H) \to [1, 3]$ with a canonical ordered bipartition (X, Y) of $V(H)$. Without loss of generality we may suppose that $x \in X$ and $y \in Y$. Then there is a canonical nd-colouring $\psi : E(G) \to [1, 3]$ that is a continuation of φ and has the canonical ordered bipartition (X, Y) of $V(G) = V(H)$.

Namely, if $S_\varphi(x) \cap S_\varphi(y) \neq \emptyset$, using $\psi(xy) \in S_\varphi(x) \cap S_\varphi(y)$ leads to $S_\psi(x) = S_\varphi(x)$ and $S_\psi(y) = S_\varphi(y)$.

If $S_\varphi(x) \cap S_\varphi(y) = \emptyset$, there is $i \in [1, 2]$ such that $S_\varphi(x) = \{i\}$ and $S_\varphi(y) = \{3\}$; in such a case setting $\psi(xy) := 3$ yields $S_\psi(x) = \{i, 3\} \in \mathcal{S}_1$ and $S_\psi(y) = \{3\} \in \mathcal{S}_2$. $\square$

3 Main result

We use the following lemma proved in [1].

Lemma 7. *If G is a graph having neither K_2 nor K_3 as a component, then G can be written as an edge-disjoint union of $\lceil \log_2 \chi(G) \rceil$ bipartite graphs, each of which has no component K_2.*

Theorem 8. $\text{gndi}(G) \leq 2\lceil \log_2 \chi(G) \rceil + 1$ *for any graph G without isolated edges.*

Proof. Because of Proposition 1 we may suppose without loss of generality that G is connected. If $G = K_1$, then $\text{gndi}(G) = 0$. For $G = K_3 = C_3$ Proposition 4 yields $\text{gndi}(G) = 3$. If $G \notin \{K_1, K_3\}$, put $r := \lceil \log_2 \chi(G) \rceil$. By Lemma 7 we know that G can be written as an edge-disjoint union of r bipartite graphs, each of which has no component K_2. Let $B_1, \dots, B_r$ be such an edge-disjoint decomposition of G. By Theorem 6, for any $i \in [1, r]$

there is an ordered bipartition (X_i, Y_i) of $V(B_i)$ and a colouring $\varphi_i : E(B_i) \to \{1, 2i, 2i+1\}$ satisfying $S_{\varphi_i}(x) \in \{\{2i\}, \{2i+1\}, \{1, 2i\}, \{1, 2i+1\}\}$ for every $x \in X_i$ and $S_{\varphi_i}(y) \in \{\{1\}, \{2i, 2i+1\}$ for every $y \in Y_i$. Then $\varphi := \bigcup_{i=1}^{r} \varphi_i$, the common continuation of all φ_i's, is an nd-colouring of G. Indeed, for any edge $e \in E(G)$ there is $i \in [1, r]$ such that $e \in E(B_i)$, and so $e = xy$ with $x \in X_i$ and $\in Y_i$. Trivially, $S_{\varphi_i}(x) \subseteq S_\varphi(x)$ and $S_{\varphi_i}(y) \subseteq S_\varphi(y)$. Since exactly one of the colours $2i, 2i+1$ is in $S_\varphi(x)$ and $S_\varphi(y)$ contains either both colours $2i, 2i+1$ or none of them, we have $S_\varphi(x) \neq S_\varphi(y)$. Thus, the colouring $\varphi : E(G) \to [1, 2r+1]$ shows that $\mathrm{gndi}(G) \leq 2r + 1 = 2\lceil \log_2 \chi(G) \rceil + 1$. □

Corollary 9. $\mathrm{gndi}(G) \leq 5$ *for any planar graph* G *without isolated edges.* □

It may be a little bit surprising that $\mathrm{gndi}(I) = 3$ for the icosahedron graph I. In fact, we do not know any planar graph whose general neighbour-distinguishing index is greater than 3.

Problem 1. *Does there exist a planar graph* G *with* $\mathrm{gndi}(G) > 3$ *?*

Theorem 10. $\mathrm{gndi}(K_n) = \lceil \log_2 n \rceil + 1$ *for any integer* $n \geq 3$*.*

Proof. In an nd-colouring of K_n any two distinct vertices must have distinct colour sets. So, using the result of [3] mentioned in the Introduction, $\mathrm{gndi}(K_n) = \chi_0(K_n) = \lceil \log_2 n \rceil + 1$. □

Corollary 11. $\mathrm{gndi}(G) \leq 2\lceil \log_2 \Delta(G) \rceil + 1$ *for any connected graph* $G \neq K_2$*.*

Proof. If there is $n \geq 3$ such that $G \simeq C_n$ or $G \simeq K_n$, use Proposition 4 or Theorem 10, respectively. Otherwise, by Brooks' Theorem, $\chi(G) \leq \Delta(G)$, and the statement follows from Theorem 8. □

As Propositions 3 and 4 show, there are 2-chromatic graphs G satisfying $\mathrm{gndi}(G) = 2\lceil \log_2 \chi(G) \rceil + 1$. However, we have been unable to find even a graph H with $\chi(H) > 2$ and $\mathrm{gndi}(H) > \lceil \log_2 \chi(H) \rceil + 1$.

Problem 2. *Find a sharp upper bound for* $\mathrm{gndi}(G)$ *as a function of* $\chi(G)$*.*

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