Irregular Labellings of Circulant Graphs

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Abstract

We investigate the *irregularity strength* ($\mathrm{s}(G)$) and *total vertex irregularity strength* ($\mathrm{tvs}(G)$) of circulant graphs $Ci_n(1, 2, \ldots, k)$ and prove that $\mathrm{tvs}(Ci_n(1, 2, \ldots, k)) = \left\lceil \frac{n+2k}{2k+1} \right\rceil$, while $\mathrm{s}(Ci_n(1, 2, \ldots, k)) = \left\lceil \frac{n+2k-1}{2k} \right\rceil$ except if either $n = 2k + 1$ or if k is odd and $n \equiv 2k + 1 (\mathrm{mod} 4k)$, then $\mathrm{s}(Ci_n(1, 2, \ldots, k)) = \left\lceil \frac{n+2k-1}{2k} \right\rceil + 1$.

Keywords: Irregularity strength, total vertex irregularity strength, graph weighting, graph labelling, circulant graph

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1. Introduction

Let us consider a simple undirected graph $G = (V(G), E(G))$ without loops, without isolated edges and with at most one isolated vertex. We assign a label $w(e)$ (called also weight), being a positive integer, to every edge $e \in E(G)$. For every vertex $v \in V(G)$ we define its *weighted degree* as

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$$wd(v) = \sum_{e \ni v} w(e).$$

We call weighting a w *irregular* if for each pair of vertices, their weighted degrees are distinct. In [8] the authors defined the graph parameter $\mathrm{s}(G)$ called the *irregularity strength* of G being the smallest integer s such that there exists an irregular weighting of G with integers $\{1, 2, \ldots, s\}$. The value of $\mathrm{s}(G)$ is known only for some special classes of graphs, e.g. complete graphs ([8]), graphs with the components being paths and cycles ([2],[11]), or some families of trees ([3],[14]).

The lower bound on the $\mathrm{s}(G)$ is given by the inequality

$$\mathrm{s}(G) \geq \max_{1 \leq i \leq \Delta} \frac{n_i + i - 1}{i}. \tag{1}$$

In the case of d-regular graphs it reduces to

$$\mathrm{s}(G) \geq \frac{n + d - 1}{d}. \tag{2}$$

The conjecture stated in [8] says that the value of $\mathrm{s}(G)$ is for every graph equal to the above lower bound plus some constant not depending on G. The first upper bounds including the vertex degrees in the denominator were given in [9] (cn/δ with relatively large values of c, depending on the relation between n, δ and Δ), then improved (slight reduction of c) in [12] and [13]. The best upper bounds known so far can be found in [10]. Namely, the authors have proved that

$$\mathrm{s}(G) \leq \left\lceil \frac{6n}{\delta} \right\rceil. \tag{3}$$

The following variant of irregularity strength that allows the vertices to be labeled as well was introduced in [6]. Now the weighted degree is defined as

$$wd(v) = \sum_{e \ni v} w(e) + w(v).$$

The corresponding graph parameter, $\mathrm{tvs}(G)$, is called *total vertex irregularity strength*. The authors of [6] gave the following lower and upper bounds:

$$\left\lceil \frac{n + \delta(G)}{\Delta(G) + 1} \right\rceil \leq \mathrm{tvs}(G) \leq n + \Delta(G) - 2\delta(G) + 1. \tag{4}$$

In the case of d-regular graphs this reduces to

$$\left\lceil \frac{n + d}{d + 1} \right\rceil \leq \mathrm{tvs}(G) \leq n - d + 1. \tag{5}$$

The exact values of $\mathrm{tvs}(G)$ are known only for a few families of graphs, e.g. complete graphs, paths and cycles ([6]) or forests without vertices of degree 2 ([5]). The best upper bound on $\mathrm{tvs}(G)$ is given in [4]:

$$\mathrm{tvs}(G) \leq \left\lceil \frac{3n}{\delta} \right\rceil + 1. \tag{6}$$

Let us consider circulant graphs defined as follows (see e.g. [7]).

Definition 1.1. *Let n and $s_1, s_2, \ldots, s_k$ be integers, with $1 \leq s_1 < \cdots < s_k \leq n/2$. The circulant graph $G = Ci_n(s_1, \ldots, s_k)$ of order n is a graph with vertex set $V(G) = \{0, 1, \ldots, n-1\}$ and edge set $E(G) = \{(x, x \pm s_i \mod n), x \in V(G), 1 \leq i \leq k\}$.*

Note that $Ci_n(s_1, \ldots, s_k)$ is $2k$-regular. The main result given in [7] says that in the case $k = 2$ and $s_1 = 1$,

$$\mathrm{s}(Ci_n(1, s_2)) = \left\lceil \frac{n+3}{4} \right\rceil \tag{7}$$

if only $s_2 \geq 2$ and $n \geq 4s_2 + 1$. Observe that in this case the value $\mathrm{s}(G)$ is equal to the lower bound given by (2).

In [1] the authors gave the exact value of total vertex irregularity strength of the graphs $Ci_n(1, 2)$:

$$\mathrm{tvs}(Ci_n(1, 2)) = \left\lceil \frac{n+4}{5} \right\rceil. \tag{8}$$

In this paper we consider a more general case of circulant graphs, $Ci_n(1, 2, \ldots, k)$, i.e. the k-th powers of cycles C_n^k. We prove the two following theorems.

Theorem 1.2. *If $k \geq 2$ and $n \geq 2k + 1$, then*

$$\mathrm{tvs}(C_n^k) = \left\lceil \frac{n+2k}{2k+1} \right\rceil.$$

Theorem 1.3. *If $k \geq 2$ and $n \geq 2k + 1$, then*

$$\mathrm{s}(C_n^k) = \begin{cases} \lceil \frac{n+2k-1}{2k} \rceil + 1, & \textit{if } n = 2k+1 \textit{ or if } n \equiv 2k+1 (\mathrm{mod} 4k) \textit{ and } k \textit{ is odd,} \\ \lceil \frac{n+2k-1}{2k} \rceil, & \textit{otherwise.} \end{cases}$$

2. Proof of Theorem 1.2

The lower bound on $\mathrm{tvs}(G)$ is given by (5), we now present an upper bound. For simplicity write $\lceil \frac{n+2k}{2k+1} \rceil = \lceil \frac{n-1}{2k+1} \rceil + 1 = s + 1$. Further, observe that because C_n^k is a regular graph it is enough to find an irregular weighting with weights $\{0, 1, 2, \ldots, s\}$ (to complete the final weighting just add 1 to every edge and vertex label). Finally, throughout the proof if some edge or vertex has not been assigned a label explicitly, then we assume it is assigned label 0. When we speak of a set of vertices being *consecutive* we mean in the initial cycle C_n. Similarly, the *distance* of two vertices refers to their distance in the original cycle C_n.

Lemma 2.1. *For any induced subgraph S of $2t+1 \leq 2k+1$ consecutive vertices $v_0, v_1, \ldots, v_{2t}$ of C_n^k there is a labeling of the edges of S with weights $\{0, s\}$ such that the weighted degrees of vertices $v_0, v_1, \ldots, v_t$ are $0, s, 2s, \ldots, ts$, respectively and the weighted degrees of $v_{t+1}, v_{t+2}, \ldots, v_{2t}$ are $(2t-1)s, (2t-2)s, \ldots, ts$, respectively.*

Note that the above lemma holds for an induced subgraph of $2t$ consecutive vertices if we remove v_0 and weighted degree 0 from the statement.

Proof. Label with s all edges in S with endpoints of distance at most t in C_n and at least one endpoint among the vertices $v_{t+1}, v_{t+2}, \ldots, v_{2t}$. Then the vertices $v_0, v_1, \ldots, v_t$ clearly have weighted degrees $0, s, 2s, \ldots, ts$ and the vertices $v_{t+1}, v_{t+2}, \ldots, v_{2t}$ have weighted degree equal to s times the number of neighbors of distance $\leq t$ (along C_n) in S i.e. $(2t-1)s, (2t-2)s, \ldots, ts$. $\square$

Note that Lemma 2.1 leaves two vertices with the same weighted degree, namely ts. In the next lemma we will resolve this repetition (we are only concerned with $t = k$ here).

Lemma 2.2. *For any induced subgraph S of $4k+2$ consecutive vertices of C_n^k there is a labeling of the edges of S with weights $\{0, s\}$ such that the set of weighted degrees of the first $2k+1$ vertices and the set of weighted degrees of the last $2k+1$ vertices are both equal to $\{0, s, 2s, \ldots, 2ks\}$.*

Proof. Apply Lemma 2.1 to the first $2k+1$ vertices and to the last $2k+1$ vertices in reverse order. Now label with s each edge in S between vertices of distance k in C_n such that one endpoint is among the first $2k+1$ vertices and one endpoint is among the last $2k+1$ vertices. Thus the weighted degree of each vertex with weighted degree $(2k-1)s, (2k-2)s, \ldots, ks$ increases by s and therefore the set of weighted degrees of the first and last $2k+1$ vertices are both equal to $\{0, s, 2s, \ldots, 2ks\}$. □

We are now ready to proceed with the labeling of C_n^k. Partition C_n^k into a vertex x and $s-1$ many segments of $2k+1$ consecutive vertices (call them $S_1, S_2, \ldots, S_{s-1}$) and one segment of $\leq 2k+1$ vertices (called S_0) such that the vertex x is between S_0 and S_{s-1}. Such a partition follows from $\lceil \frac{n-1}{2k+1} \rceil + 1 = s+1$. Let us pair the segments as follows: $(S_1, S_2), (S_3, S_4), \ldots, (S_{r-1}, S_r)$ where $r \in \{s-1, s-2\}$ (depending on the parity of s) and label the edges of each pair of segments as in Lemma 2.2.

Next for $1 \leq i \leq r$ label all vertices in S_i with i. Now the vertices of $S_1, S_2, \ldots, S_r$ have distinct weighted degrees; each of the form $ls+i$ where $i < s$. Indeed any two vertices within the same segment S_i have weighted degree $l_1 s + i$ and $l_2 s + i$ where $l_1 \neq l_2$ by Lemma 2.2 and any two vertices in different segments S_i and S_j have weighted degrees $l_1 s + i \neq l_2 s + j$. Note that no vertex of $S_1, S_2, \ldots, S_r$ has weighted degree equal to a multiple of s.

Now we must consider the remaining vertex x and segment(s) S_0 and (if s is even) S_{s-1}. We distinguish two cases.

Case 1. s is odd.

Then we must complete the labeling of x and S_0. Let $0 < |S_0| \in \{2t, 2t+1\}$ where $t \leq k$. For simplicity let us assume that $|S_0| = 2t+1$, but note that the case $|S_0| = 2t$ is essentially identical. Let $v_0, v_1, \ldots, v_{2t}$ be the vertices of S_0 such that x and v_{2t} are consecutive. Now label the edges of S_0 according to Lemma 2.1 and label the edges $xv_{2t}, xv_{2t-1}, \ldots, xv_{t+1}$ with s. Now the vertices of S_0 have weighted degrees $\{0, s, 2s, \ldots, 2ts\}$ and x has weighted degree ts. Finally label with 0 each vertex in S_0 that has weighted degree $< ts$ and label with s each vertex in S_0 that has weighted degree $\geq ts$ (the vertex x gets label 0). Now among the vertices $S_0 \cup \{x\}$ we have weighted degrees $\{0, s, 2s, \ldots, 2ts, (2t+1)s\}$ thus the vertices among $S_0 \cup \{x\}$ are distinguished from each other. The vertices of $S_0 \cup \{x\}$ have weighted degrees that are multiples of s and thus are distinguished from the remainder of the graph. (To complete the proof for $|S_0| = 2t$ merely remove vertex v_0 and weighted degree 0 from the above argument.)

Case 2. s is even.

Then we must complete the labeling of x and S_0 and S_{s-1}. Let $0 < |S_0| \in \{2t, 2t+1\}$ where $t \leq k$. As before let us assume that $|S_0| = 2t+1$, but note that the case $|S_0| = 2t$ is essentially identical. Let $v_0, v_1, \ldots, v_{2t}$ be the vertices of S_0 and let $u_0, u_1, \ldots, u_{2k}$ be the vertices of S_{s-1} such that u_{2k}, x and v_{2t} are consecutive. To complete the labeling of S_{s-1} label the edges of S_{s-1} according to Lemma 2.1 and label the edges $xu_{2k}, xu_{2k-1}, \ldots, xu_{k+1}$ with label s. Now the vertices of S_{s-1} have weighted degrees $\{0, s, 2s, \ldots, 2ks\}$ and x has weight ks. Finally label the vertices of S_{s-1} with $s-1$. Now as before the vertices of $S_1, S_2, \ldots, S_{s-1}$ have distinct weighted degrees. Furthermore, note that no vertex of $S_1, S_2, \ldots, S_{s-1}$ has weighted degree equal to a multiple of s.

To complete the labeling of S_0 label the edges of S_0 according to Lemma 2.1 and label the edges $xv_{2t}, xv_{2t-1}, \ldots, xv_{t+1}$ with label s. Now the vertices of S_0 have weighted degrees $\{0, s, 2s, \ldots, 2ts\}$ and the weighted degree of

x increases by ts to $(k+t)s$. Finally label each vertex in S_0 with 0 and label x with s. Now among the vertices of S_0 we have weighted degrees $\{0, s, 2s, \ldots, 2ts\}$ and x has weighted degree $(k+t+1)s > 2ts$. The vertices of $S_0 \cup \{x\}$ have weighted degrees that are multiples of s and thus are distinguished from the remainder of the graph. (As in the previous case, to complete the proof for $|S_0| = 2t$ merely remove vertex v_0 and weighted degree 0 from the above argument.) □

3. Proof of Theorem 1.3

For simplicity write $\lceil \frac{n+2k-1}{2k} \rceil = \lceil \frac{n-1}{2k} \rceil + 1 = s + 1$. First note that if $n = 2k+1$ then C_n^k is a complete graph and $\mathrm{s}(C_n^k) = 3$ (see Chartrand et al. [8]). Thus from here on we may assume that $s > 1$. Depending on the two cases stated in Theorem 1.3 we will find a labeling with $\{0, 1, 2, \ldots, s\}$ or $\{0, 1, 2, \ldots, s+1\}$. As in the proof of Theorem 1.2 because C_n^k is a regular graph we may obtain the final weighting by adding 1 to every edge label to get a labeling with positive integers. As before if some edge has not been assigned a label explicitly, then we assume it is assigned label 0. When we speak of a set of vertices being *consecutive* we mean in the initial cycle C_n. Similarly, the *distance* of two vertices refers to their distance in the original cycle C_n.

First let us show that at least $s+2$ labels are necessary in the case $n \equiv 2k+1 \pmod{4k}$ and k is odd. Observe that $n \equiv 2k+1 \pmod{4k}$ implies s is odd and $n = 2ks + 1$. Assume (to the contrary) that there is an irregular weighting with labels $\{0, 1, 2, \ldots, s\}$. Note that there are $2ks+1$ possible different weighted degrees (i.e. $\{0, 1, 2, \ldots, 2ks\}$) thus each weighted degree must appear exactly once. The sum of all these weighted degrees is

$2ks(2ks+1)/2 = 2k^2s^2 + ks$ is odd. However, observe that the sum of all weighted degrees must be even as each edge label is counted twice. This is a contradiction and thus at least $s+2$ labels are necessary. This fact combined with (2) gives the appropriate lower bounds for Theorem 1.3. Now we find corresponding upper bounds. First we distinguish two cases based on the parity of s.

Case 1. s is even.

Lemma 3.1. *For any induced subgraph S of $2k$ consecutive vertices $v_1, v_2, \ldots, v_{2k}$ of C_n^k and a weight $0 \le c \le s$, there is a labeling of the edges of S with weights $\{0, c, s\}$ such that the weighted degrees of vertices $v_1, v_2, \ldots, v_k$ are $c, s+c, 2s+c, \ldots, (k-1)s+c$, respectively and the weighted degrees of $v_{k+1}, v_{k+2}, \ldots, v_{2k}$ are $(2k-2)s+c, (2k-3)s+c, \ldots, (k-1)s+c$, respectively.*

Proof. First for each $i \in \{1, 2, \ldots, k\}$ label the edge v_iv_{i+k} with weight c. Next label with s all unlabeled edges with at least one endpoint among $v_{k+1}, v_{k+2}, \ldots, v_{2k}$. Now for $i \in \{1, 2, \ldots, k\}$ the vertex v_i has i neighbors in $v_{k+1}, v_{k+2}, \ldots, v_{2k}$ and thus has weighted degree $(i-1)s + c$. Similarly for $i \in \{1, 2, \ldots, k\}$ the vertex v_{k+i} has $2k - i$ neighbors in S and thus has weighted degree $(2k - i - 1)s + c$. □

Note that Lemma 3.1 leaves two vertices with the same weighted degree, namely $(k-1)s + c$. In the next lemma we will resolve this repetition.

Lemma 3.2. *For any induced subgraph S of $4k$ consecutive vertices of C_n^k and a weight $0 \le b \le s-1$, there is a labeling of the edges of S with weights $\{0, b, b+1, s\}$ such that the set of weighted degrees of the first $2k$ vertices is* *equal to* $\{b, s+b, 2s+b, \ldots, (2k-1)s+b\}$ *and the set of weighted degrees of the last* $2k$ *vertices is equal to* $\{b+1, s+b+1, 2s+b+1, \ldots, (2k-1)s+b+1\}$.

Proof. Let $v_1, v_2, \ldots, v_{4k}$ be the vertices of S. First label the subgraph induced by the first $2k$ vertices $v_1, v_2, \ldots, v_{2k}$ as in Lemma 3.1 with $c = b$. Next use Lemma 3.1 with $c = b+1$ to label the subgraph induced by the last $2k$ vertices (in reverse order) $v_{4k}, v_{4k-1}, \ldots, v_{2k+1}$. Finally, label with s the edges $v_{k+1}v_{2k+1}, v_{k+2}v_{2k+2}, \ldots, v_{2k}v_{3k}$ thus increasing the weighted degree of the middle $2k$ vertices each by s. □

Now we are ready to label C_n^k for s even. Partition C_n^k into a vertex x and $s-1$ segments $S_1, S_2, \ldots, S_{s-1}$ each of $2k$ consecutive vertices and one segment R of $1 \leq |R| \leq 2k$ consecutive vertices. For $i \in \{1, 3, 5, \ldots, s-3\}$ pair up the segments S_i and S_{i+1} and label the edges according to Lemma 3.2 with $b = i$. Clearly the weighted degrees in $S_1 \cup S_2 \cup \cdots \cup S_{s-2}$ are all distinguished (note that if $s = 2$ nothing has been done yet).

We now turn our attention to S_{s-1}, R and x. If $|R| = 2k$, then assume R and S_{s-1} are adjacent and label their edges according to Lemma 3.2 with $b = s-1$. The vertex x is left with weighted degree 0 and all vertices now have unique weighted degree.

We can now assume $|R| < 2k$. Thus $|R| \in \{2t, 2t+1\}$ and $t < k$. Assume that x is between S_{s-1} and R in C_n^k. Now apply Lemma 3.1 with $c = s-1$ to S_{s-1} such that the vertices of weighted degree $(2k-2)s+(s-1), (2k-3)s+(s-1), \ldots, (k-1)s+(s-1)$ are adjacent to x. Next label with s the edges between x and S_{s-1} and thus the vertices of S_{s-1} are distinguished and the weighted degree of x is ks.

Let $u_1, u_2, \ldots, u_t \in R$ be vertices of distance $1, 2, \ldots, t$ from x in C_n. Now weight with s all edges in R with endpoints of distance at most t in C_n

such that one endpoint is in $u_1, u_2, \ldots, u_t$ (as in Lemma 2.1 from the proof of Theorem 1.2) and label with s the edges between x and $u_1, u_2, \ldots, u_t$. Now the weighted degree of x is $(k+t)s > 2ts$ and the weighted degrees in R are $\subset \{0, s, 2s, \ldots, 2ts\}$ (the weighted degree 0 appears if $|R|$ is odd) and the weighted degrees in S_{s-1} are $\{(s-1), s+(s-1), 2s+(s-1), \ldots, 2ks+(s-1)\}$. Thus each vertex in C_n^k has a unique weighted degree.

Case 2. s is odd.

Lemma 3.3. *For any induced subgraph S of $2k$ consecutive vertices $v_1, v_2, \ldots, v_{2k}$ of C_n^k and any $0 \leq b \leq s-2$, there is a labeling of the edges of S with weights $\{0, b, s-1\}$ such that the weighted degree of each vertex $v_1, v_2, \ldots, v_k$ is $(k-1)(s-1)+b$ and the weighted degree of each vertex $v_{k+1}, v_{k+2}, \ldots, v_{2k}$ is b.*

Proof. First for each $i \in \{1, 2, \ldots, k\}$ label the edge $v_i v_{i+k}$ with weight b. Then label with $s-1$ each edge in the complete graph induced by the vertices $v_1, v_2, \ldots, v_k$. □

Now we are ready to label C_n^k for s odd. First partition C_n^k into $s-1$ segments $S_0, S_1, \ldots, S_{s-2}$ each of $2k$ consecutive vertices and one segment R of $2 \leq |R| \leq 2k+1$ consecutive vertices.

First let us weight the edges in S_0 according to Lemma 3.3 with $b = 0$. Next, for $i \in \{1, 2, \ldots, s-2\}$, let $u_1, u_2, \ldots, u_{2k}$ and $v_1, v_2, \ldots, v_{2k}$ be the vertices of S_{i-1} and S_i respectively. We distinguish two cases based on the parity of i.

If i is odd, apply Lemma 3.3 with $b = i$ to S_i where the vertices of S_i are considered in reverse order. Next label with $s-1$ all edges with one endpoint in S_i and one endpoint in S_{i-1} such that the two endpoints have distance at most $k-1$ in C_n.

If i is even, apply Lemma 3.3 with $b = i$ to S_i. Next label with $s-1$ all edges with one endpoint in S_i and one endpoint in S_{i-1}.

Finally label with $s-1$ all edges with at least one endpoint in R and all edges between S_0 and S_{s-2} (i.e. if $|R| < k$). Thus the weighted degrees of the first k vertices of S_0 and the last k vertices of S_{s-2} increase by $(s-1), 2(s-1), \ldots, k(s-1)$. Observe that at this point for $i \in \{0, 1, \ldots, s-2\}$ the set of weighted degrees of S_i is $\{i, (s-1)+i, 2(s-1)+i, \ldots, (2k-1)(s-1)+i\}$ i.e. all vertices in $C_n^k \setminus R$ are distinguished. Also, observe that the weighted degree of every vertex in R is $2k(s-1)$ and the largest weighted degree in $C_n^k \setminus R$ is $(2k-1)(s-1)+(s-2) = 2k(s-1)-1 < 2k(s-1)$ and it is the weighted degree of the last vertex of S_{s-2}.

The size of R can be written as $2t$ or $2t+1$ where $0 < t \leq k$. We will distinguish two cases depending on the parity of t.

Case 2.1 t is even.

Let A be t consecutive vertices in R and let B be the remaining t or $t+1$ consecutive vertices. Now relabel with s all edges in R with two endpoints of distance at most t in C_n such that at least one endpoint is in B. This increases the weighted degree in A by $1, 2, \ldots, t$ and in B by $|B|+t-1, |B|+t-2, \ldots, t$. Now relabel with $s-2$ the edges of any perfect matching in A. This decreases the weighted degree of each of the vertices in A by 1. Thus the weighted degrees of the vertices in R become $2k(s-1), 2k(s-1)+1, \ldots, 2k(s-1)+|R|-1$ and are therefore distinguished from each other and the rest of the graph.

Case 2.2 t is odd.

If $|R| = 2k+1$, then $k = t$ is odd and we must use label $s+1$ in addition to labels $\{0, 1, 2, \ldots, s\}$. At this moment all edges in R are labeled with $s-1$ and all weighted degrees are $2k(s-1)$. Let A be $k+1$ consecutive vertices in R and let B be the remaining k consecutive vertices. Relabel with s every edge with an endpoint among A thus increasing the weighted degrees in A by $k, k+1, \ldots, 2k$ and in B by $k, k-1, \ldots, 1$. Finally, relabel with $s+1$ the edges of any perfect matching in A. This increases each weighted degree in A by 1. Now all weighted degrees in R are unique and each is greater than $2k(s-1)$ and thus all weighted degrees in C_n^k are unique.

From here we may assume $|R| \leq 2k$. Now we distinguish two cases based on the size of R.

Case 2.2.1 $|R| = 2t$.

Let A be $t-1$ (possibly 0) consecutive vertices in R and let B be the remaining $t+1$ consecutive vertices. Furthermore, let w be the vertex of S_{s-2} that is distance at most $t-1$ from all vertices in A and put $A' = A \cup \{w\}$. Note that the weighted degree of w is $2k(s-1)-1$.

Now relabel with s all edges with both endpoints in B and all edges with one endpoint in B and one endpoint in A' such that the two endpoints have distance at most t in C_n. This increases the weighted degree of w by 1 and the vertices in A by $2, 3, \ldots, t$ and the vertices in B by $2t, 2t-1, \ldots, t$. Now relabel with $s-2$ the edges of any perfect matching in A. This decreases the weighted degree of each of the vertices in A by 1. Thus the weighted degree of w is $2k(s-1)$ and the weighted degrees in R are $2k(s-1)+1, 2k(s-1)+2, \ldots, 2k(s-1)+2t$ and are therefore distinguished from each other and the rest of the graph.

Case 2.2.2 $|R| = 2t+1$.

Then $t < k$ as $|R| \leq 2k$. Let A be t consecutive vertices in R and let B be the remaining $t+1$ consecutive vertices. Furthermore, let w be the vertex of S_{s-2} that is distance at most t from all vertices in A and put $A' = A \cup \{w\}$.

Now relabel with s all edges with both endpoints in B and all edges with one endpoint in B and one endpoint in A' such that the two endpoints have distance at most $t+1$ (as $t < k$) in C_n. This increases the weighted degree of w by 1 and the vertices in A by $2, 3, \dots, t+1$ and the vertices in B by $2t+1, 2t, \dots, t+1$. Now relabel with $s-2$ the edges of any perfect matching in A'. This decreases the weighted degree of each of the vertices in A' by 1. Thus finally the weighted degree of w remains unchanged and the weighted degrees in R are $2k(s-1)+1, 2k(s-1)+2, \dots, 2k(s-1)+2t+1$ and are therefore distinguished from each other and the rest of the graph. $\square$

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