The unbalance of set systems

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Abstract

The unbalance of an intersecting family $\mathcal{F}$ is defined as $|\mathcal{F}| - d(\mathcal{F})$, where $d(\mathcal{F})$ is the maximum degree of $\mathcal{F}$; the max $|\{F \in \mathcal{F} : x \in F\}|$ over all vertices x. We show that the unbalance of a k-uniform intersecting family is at most $\binom{n-3}{k-2}$ when $n \geq 6k^3$ and we determine all families achieving this bound.

1 Introduction

Let X be a set of n elements. A family of sets $\mathcal{F} \subset \binom{X}{k}$ is *intersecting* if for any sets $F, G \in \mathcal{F} \Rightarrow F \cap G \neq \emptyset$. We denote the degree of a vertex, x, by $d(x) = |\{F \in \mathcal{F} : x \in F\}|$. Similarly, $d(xy)$ denotes the number of sets containing x and y, so $d(xy) = |\{F \in \mathcal{F} : x, y \in F\}|$. We denote the maximum degree in $\mathcal{F}$ by $d(\mathcal{F})$. The *unbalance*, $u(\mathcal{F})$, of an intersecting family $\mathcal{F}$ is $|\mathcal{F}| - d(\mathcal{F})$. A *transversal* of a family $\mathcal{F}$ is a set $T \subseteq X$ such that $\forall\, F \in \mathcal{F}$, $F \cap T \neq \emptyset$. For $\mathcal{F}$ intersecting, such a set always exists because any $F \in \mathcal{F}$ is trivially a transversal for $\mathcal{F}$. From now on when we speak of a

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transversal we assume it is of minimal size. This value is denoted $\tau(\mathcal{F})$ and is called the *transversal number* of the family.

In [6], Dinur and Friedgut proved, among other things, the following for finite intersecting families.

Proposition 1. *If $\mathcal{F} \subset \binom{X}{k}$ is intersecting and $k < \frac{n}{2}$ then all members of $\mathcal{F}$, with the exception of $O(\binom{n-2}{k-2})$ contain some fixed element $x \in X$.*

Motivated by this statement, Gyula O. H. Katona suggested that the maximum unbalance of a k-uniform intersecting family should be determined.

In this paper we find an upper bound of the unbalance for $n > n_0(k)$ and prove it is sharp. Furthermore, we describe every possible family achieving the maximal unbalance. The idea of classifying famlies based on their transversal number plays heavily in our proof. In Section 2 we provide some constructions achieving the upper bound on the unbalance, in Section 3 we give a simple proof of the upper bound, and in Section 4 we expand our proof by examining the case when the transversal number is 4 or greater to improve the threshold $n_0(k)$ obtained in Section 3.

2 Constructions

We begin with the celebrated result of Erdös, Ko and Rado [1].

Theorem 2. *If $\mathcal{F} \subset \binom{X}{k}$ is intersecting and $n > 2k$ then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Equality holds if and only if there exists some $x \in X$ such that we have $\mathcal{F} = \{F \subset X : |F| = k, x \in F\}$.* □

We also remark that the bound in Theorem 2 is true when $n = 2k$, but for $n < 2k$ we can consider the family of all sets of size k which is necessarily intersecting and has $|\mathcal{F}| \geq \binom{n-1}{k-1}$.

Observe that in the case of equality in Theorem 2 we have a family with unbalance equal to 0. Families with transversal number equal to 1 are referred to as *trivial* families. Hilton and Milner [2] proved that the largest nontrivial k-uniform intersecting family has size $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ when $n \geq k$. It is worth noting that this family has only slightly larger unbalance; it has an unbalance of 1.

We will use the following definition for our constructions. A $(2l+1)$-*kernel system* is a k-uniform family with the following construction. Take a $(2l+1)$-element subset of the underlying set X. This is the *kernel* of the family. The family consists of all k-sets containing at least $(l+1)$ vertices from the kernel. It is clear this family is intersecting.

The 3-kernel system has size $3\binom{n-3}{k-2}+\binom{n-3}{k-3}$. The maximum degree of the 3-kernel system is $2\binom{n-3}{k-2}+\binom{n-3}{k-3}$ because a vertex in the kernel is in all sets containing exactly 3 kernel elements and in $\frac{2}{3}$ of all sets containing exactly 2 kernel elements. Therefore the 3-kernel system has unbalance $\binom{n-3}{k-2}$. We will show that this is the maximal unbalance a k-uniform intersecting family can achieve. In fact, we will show that every family achieving this bound is either isomorphic to the 3-kernel system or to a sub-family of the 3-kernel system which contains all the k-sets intersecting the kernel in exactly 2 elements. We prove that this is true for $n > 6k^3$.

3 Main result

Theorem 3. *If $\mathcal{F} \subset \binom{X}{k}$ is intersecting and $n > n_0(k) = 6k^3$ then the unbalance of $\mathcal{F}$ is $u(\mathcal{F}) \leq \binom{n-3}{k-2}$.*

First we prove Theorem 3 with a weaker threshold $n_0(k)$ by distinguishing the cases $\tau(\mathcal{F}) = 1$, $\tau(\mathcal{F}) = 2$, and $\tau(\mathcal{F}) \geq 3$. The case $\tau(\mathcal{F}) = 2$ will give us our 3-kernel system. For the case $\tau(\mathcal{F}) \geq 3$, we merely estimate the size of the family which necessarily gives an upper bound on the unbalance. In Section 4 we split the case $\tau(\mathcal{F}) \geq 3$ into $\tau(\mathcal{F}) = 3$ and $\tau(\mathcal{F}) \geq 4$ to improve $n_0(k)$ to the desired value. Frankl [3] gives the exact bounds on the size of families with $\tau(\mathcal{F}) = 3$. Similarly, Frankl, Ota, and Tokushige [4] find the exact bounds on the size of families with $\tau(\mathcal{F}) = 4$. However, we include our calculations here both for the sake of completeness (as the calculations are fairly simple) and because our results do not require n to depend on $\tau(\mathcal{F})$. This is important as we will show that Theorem 3 is true for all $n > 6k^3$.

Proof. Let T be a minimal transversal for $\mathcal{F}$. We distinguish three cases.

1. $|T| = 1$. Then there exists a vertex x that is contained in every $F \in \mathcal{F}$. So, $d(\mathcal{F}) = d(x) = |\mathcal{F}|$ gives $u(\mathcal{F}) = 0$.

2. $|T| = 2$. Let $T = \{x, y\}$. Let $\mathcal{F}_x = \{F \in \mathcal{F} : F \cap T = \{x\}\}$. Similarly, let $\mathcal{F}_y = \{F \in \mathcal{F} : F \cap T = \{y\}\}$. Now, $|\mathcal{F}| = d(x) + d(y) - d(xy)$ and clearly

$d(\mathcal{F}) \geq \max\{d(x), d(y)\}$. This gives

$$\begin{aligned} u(\mathcal{F}) &= |\mathcal{F}| - d(\mathcal{F}) \\ &\leq d(x) + d(y) - d(xy) - \max\{d(x), d(y)\} \\ &= \min\{d(x), d(y)\} - d(xy) \\ &= \min\{|\mathcal{F}_x|, |\mathcal{F}_y|\}. \end{aligned}$$

Let $\mathcal{F}_x - x$ be the family $\mathcal{F}_x$ with x removed from each set. Clearly $|\mathcal{F}_x| = |\mathcal{F}_x - x|$. Note that $\mathcal{F}_x - x$ and $\mathcal{F}_y - y$ as $(k-1)$-uniform families on the underlying set $X \setminus \{x, y\}$. Because $\mathcal{F}$ is intersecting each set of $\mathcal{F}_x - x$ must intersect each set of $\mathcal{F}_y - y$. That is, $\mathcal{F}_x - x$ and $\mathcal{F}_y - y$ are cross-intersecting $(k-1)$-uniform families on a ground set of size $n-2$. We can now apply a theorem of Frankl and Tokushige [5]:

Theorem 4. *Suppose $\mathcal{F}, \mathcal{G} \subset \binom{X}{k}$, are cross-intersecting families with $n \geq 2k$. If $|\mathcal{F}| \geq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ and $\mathcal{F}$ is nontrivial, then $|\mathcal{G}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.* □

If at least one of $\mathcal{F}_x - x, \mathcal{F}_y - y$ is nontrivial, then we can apply this theorem directly, so that,

$$\begin{aligned} u(\mathcal{F}) &\leq \min\{|\mathcal{F}_x|, |\mathcal{F}_y|\} \\ &= \min\{|\mathcal{F}_x - x|, |\mathcal{F}_y - y|\} \\ &\leq \binom{(n-2)-1}{(k-1)-1} - \binom{(n-2)-(k-1)-1}{(k-1)-1} + 1 \\ &= \binom{n-3}{k-2} - \binom{n-k-1}{k-2} + 1 \\ &< \binom{n-3}{k-2}. \end{aligned}$$

On the other hand, if both $\mathcal{F}_x - x$ and $\mathcal{F}_y - y$ are trivial and maximal, Theorem 2 implies that there is a z_1 and z_2 contained in all the sets of $\mathcal{F}_x - x$ and $\mathcal{F}_y - y$ respectively. In this case we have $|\mathcal{F}_x| = |\mathcal{F}_y| = \binom{(n-2)-1}{(k-1)-1} = \binom{n-3}{k-2}$. Since $\mathcal{F}_x - x$ and $\mathcal{F}_y - y$ are cross-intersecting, $z_1 = z_2$ and $\mathcal{F}$ is exactly the 3-kernel system or one of its sub-families described above with kernel $\{x, y, z_1\}$.

3. $|T| \geq 3$. Let $x, y, z \in T$. By the minimality of $|T|$, for each $v \in T$, $\exists F \in \mathcal{F}$ such that $F \cap T = \{v\}$ (otherwise v does not need to be included

in T). Let $F, G \in \mathcal{F}$ such that $F \cap T = \{x\}$ and $G \cap T = \{y\}$. Let $L = F \cap G$. Now for each point $v \in L$, $\exists\, F_v \in \mathcal{F}$ such that $F_v \cap \{x, y\} \neq \emptyset$ and $v \notin F_v$ (otherwise v can replace both x and y in T contradicting the minimality of $|T|$). Now we can find a simple upper bound for $d(z)$ by overcounting the number of sets that contain z.

Consider a set H containing z. H must intersect F and G. We distinguish two subcases.

3.1. H intersects both F and G in some $v \in L$. Then H must intersect F_v. The point v can be chosen in $|L| < k$ ways. The intersection of H and F_v can be chosen in $|F_v| = k$ ways. The remaining $k-3$ points of H can be chosen in $\binom{n-3}{k-3}$ ways. This gives less than $k^2\binom{n-3}{k-3}$ choices for H.

3.2. H intersects F and G in distinct points outside of L. The intersection of H and F can be chosen in $k - |L| < k$ ways. The intersection of H and G can be chosen in $k - |L| < k$ ways. The remaining $k-3$ points of H can be chosen in $\binom{n-3}{k-3}$ ways. This gives less than $k^2\binom{n-3}{k-3}$ choices for H.

In total we have $d(z) < 2k^2\binom{n-3}{k-3}$. It should be noted that in general (for $k > 3$) this will be a gross overcount. The choice of z from T was arbitrary, so $\forall\, v \in T$, $d(v) < 2k^2\binom{n-3}{k-3}$. Clearly $|T| \leq k$, so we have $|\mathcal{F}| \leq \sum_{v \in T} d(v) < 2k^3\binom{n-3}{k-3}$. For $n > 2k^3(k-2) + k$ we have $2k^3\binom{n-3}{k-3} < \binom{n-3}{k-2}$. □

With a little more work we can improve the threshold $n_0(k)$ to $6k^3$. It is obvious that if case 3 above is restricted to when $\tau(\mathcal{F}) = 3$ then we get the bound $|\mathcal{F}| < 6k^2\binom{n-3}{k-3}$. So, $|\mathcal{F}| < \binom{n-3}{k-2}$ for $n > 6k^3 - 12k^2 + k$ Then we merely need to bound $|\mathcal{F}|$ when $\tau(\mathcal{F}) \geq 4$ and show $|\mathcal{F}| < \binom{n-3}{k-2}$ for $n > 6k^3$. This is the object of Section 4.

We do not know if $n_0(k) = O(k^3)$ is best possible. An easy lower bound for $n_0(k)$ comes from the 5-kernel system. In this case we have all k-sets with exactly 3, 4, or 5 elements from the kernel. That is, $|\mathcal{F}| = 10\binom{n-5}{k-3} + 5\binom{n-5}{k-4} + \binom{n-5}{k-5}$. If we fix a point from the kernel and count the number of sets that contain it we get the maximum degree, $d(\mathcal{F}) = 6\binom{n-5}{k-3} + 4\binom{n-5}{k-4} + \binom{n-5}{k-5}$. Therefore the unbalance is $u(\mathcal{F}) = 4\binom{n-5}{k-3} + \binom{n-5}{k-4}$. This is larger than $\binom{n-3}{k-2}$ when $n < 3k-2$. We do not know of any constructions with unbalance larger than $\binom{n-3}{k-2}$ for $3k - 2 < n < 6k^3$.

4 Transversal number at least 4

In this section we find an upper bound on the size of a k-uniform intersecting family $\mathcal{F}$ with $\tau(\mathcal{F}) \geq 4$.

Suppose $\mathcal{F}$ is intersecting with $\tau(\mathcal{F}) \geq 4$. Then there is a minimal transversal T of the family containing at least four elements. Let t_1, t_2, t_3, t_4 be four distinct elements of T. By minimality of $|T|$, there are sets F_1, F_2, F_3 in the family such that $T \cap F_i = \{t_i\}$. We will find an upper bound on the degree of t_4. Each set H containing t_4 must intersect each set F_i for $i \leq 3$. We distinguish three cases.

1. Suppose we pick a point $x \in \bigcap_{i \leq 3} F_i$ to be in H (if such an x exists). Now x cannot replace t_1, t_2, and t_3, in the transversal T (otherwise $|T|$ would not be minimal) so there must exist 3 sets G_1, G_2, G_3 meeting $\{t_1, t_2, t_3\}$ such that: $x, t_4 \notin G_i$, and $\bigcap_{i \leq 3} G_i = \emptyset$. Otherwise all sets containing t_1, t_2, and t_3, also contain at least one of x, t_4 or some point $z \in \bigcap_{i \leq 3} G_i$. But then we can replace t_1, t_2, t_3 in the transversal with x, z, t_4 contradicting the minimality of $|T|$. Now H must intersect each G_i, so we must choose a point from one of the three different pairwise intersections of G_is and one point from the remaining G_i. The remaining $k-4$ points of H we choose arbitrarily. No more than $3k^3\binom{n-4}{k-4}$ sets containing t_4 are constructed in this way.

2. Suppose we pick a point x in the intersection of exactly two of the sets F_i, say F_1 and F_2 (the two sets can be chosen in 3 ways), to be in H. Now H must intersect F_3. Let it do so in the point y. Then, as above, there must exist a set G meeting $\{t_1, t_2, t_3\}$ such that $x, y, t_4 \notin G$. Otherwise x, y, t_4 could replace t_1, t_2, t_3 in T, contradicting the minimality of $|T|$. Now H must also intersect G. We choose the remaining $k-4$ points of H arbitrarily. No more than $3k^3\binom{n-4}{k-4}$ sets containing t_4 are constructed in this way.

3. Finally suppose we pick points $x_i \in F_i \setminus F_j$ for $1 \leq i, j \leq 3$, and $j \neq i$ to be in H. We choose the remaining $k-4$ points of H arbitrarily. No more than $k^3\binom{n-4}{k-4}$ sets containing t_4 are constructed in this way.

This overcount gives $d(t_4)$ to be at most $7k^3\binom{n-4}{k-4}$. Since t_4 was chosen arbitrarily from T, $|T| \leq k$, we conclude that $|\mathcal{F}| \leq \sum_{t \in T} \leq 7k^4\binom{n-4}{k-4}$. We note that this is less than $\binom{n-3}{k-2}$ for $n > 6k^3$ as desired.

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