Introduction of pseudo-Anosov mapping classes

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$S_{g,n}$ is a genus g surface with n marked points.
$f : S_{g,n} \to S_{g,n}$ is a surface homeomorphism.

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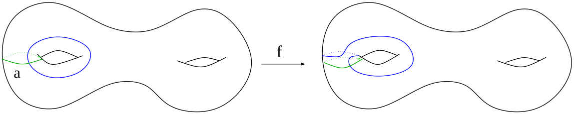
Example

$f : S_{2,0} \to S_{2,0}$ is Dehn twist along a. It's easy to check $f(a) = a$, so f is reducible.



pseudo-Anosov homeomorphisms

Definition

f is **pseudo-Anosov** if there are transverse singular foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ with transverse measure μ^s and μ^u, such that $\exists\ \lambda > 1$,

$$f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda\mu^s),$$
$$f(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda^{-1}\mu^u).$$

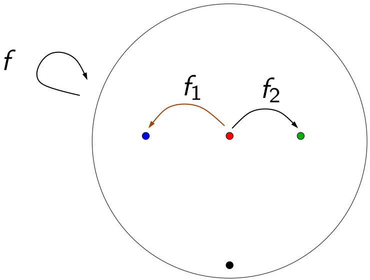
Definition

$\lambda(f) :=$ the dilatation of pseudo-Anosov f.

Example of pseudo-Anosov homeomorphisms

Example (Thurston)

$f : S_{0,4} \longrightarrow S_{0,4}$ (Red then Black) is pseudo-Anosov.



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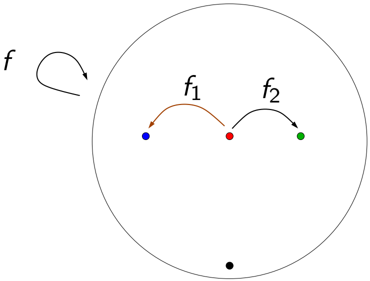

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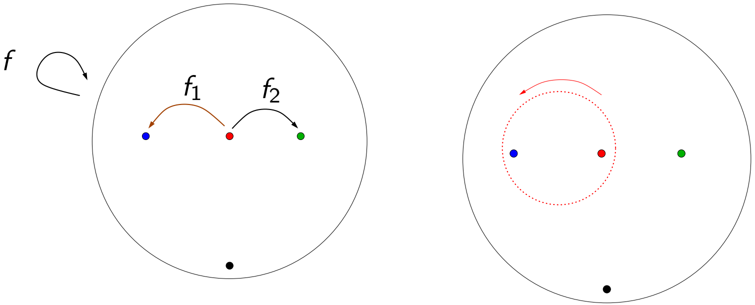

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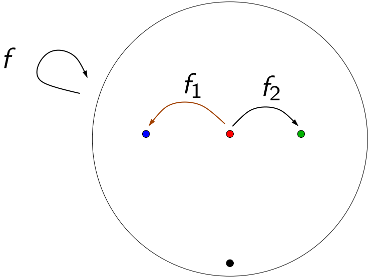

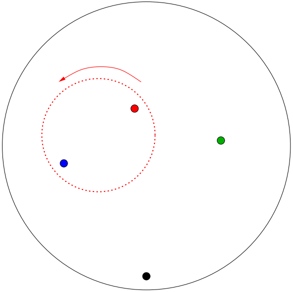

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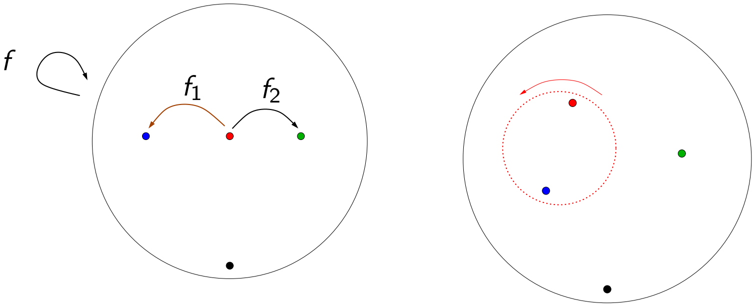

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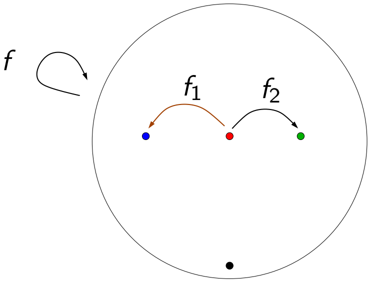

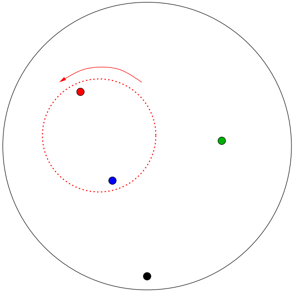

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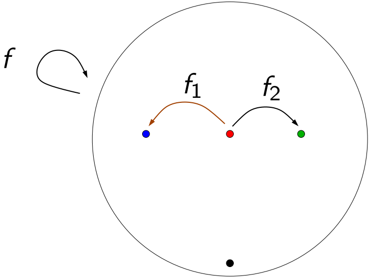

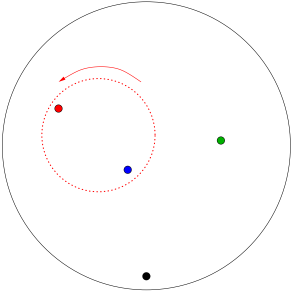

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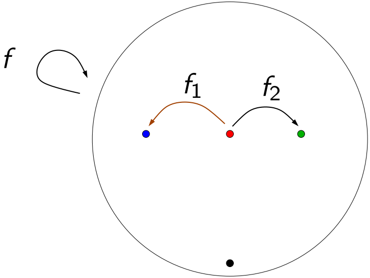

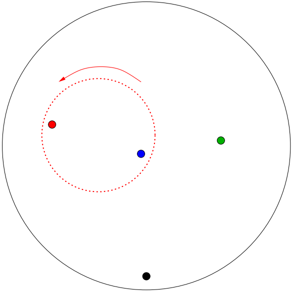

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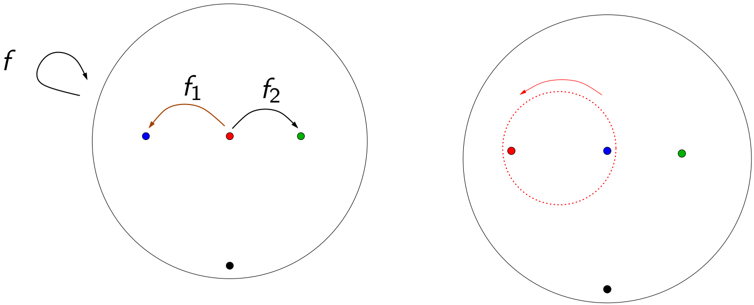

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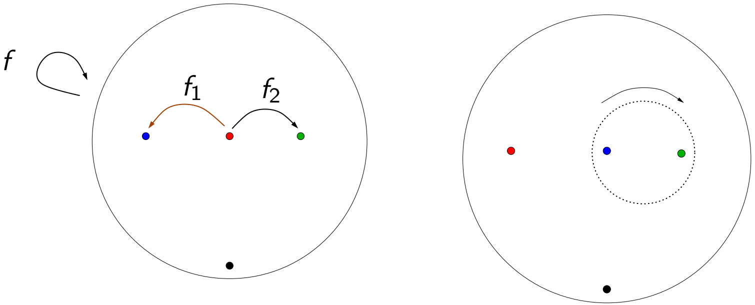

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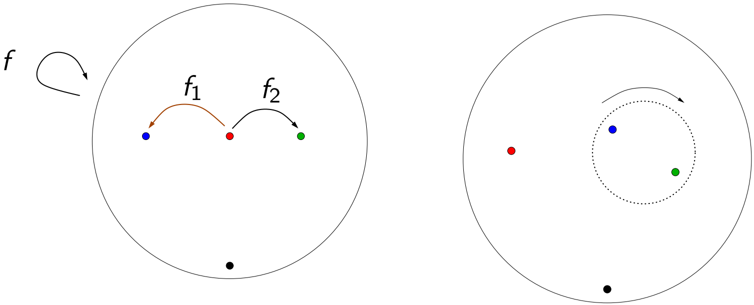

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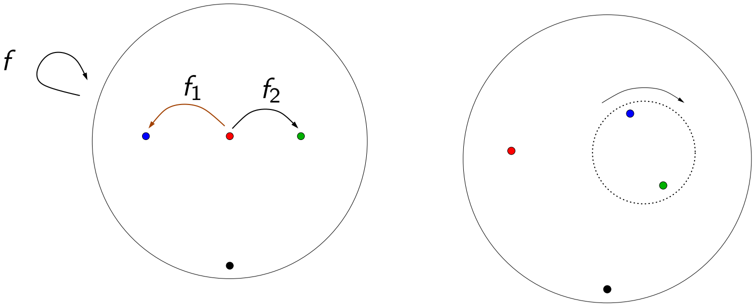

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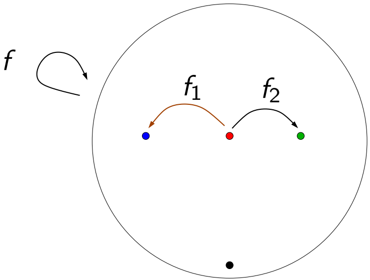

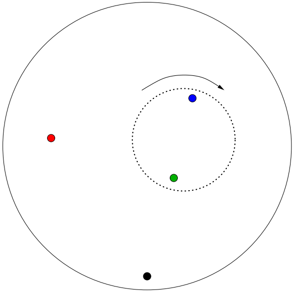

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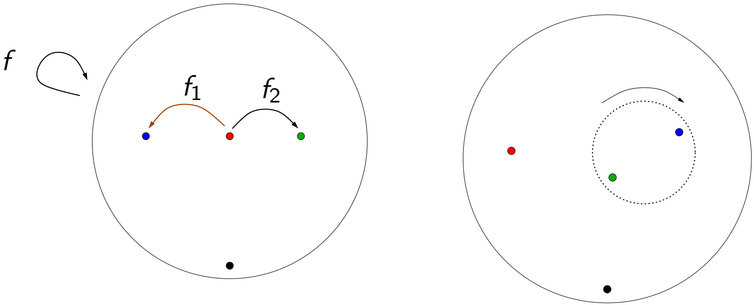

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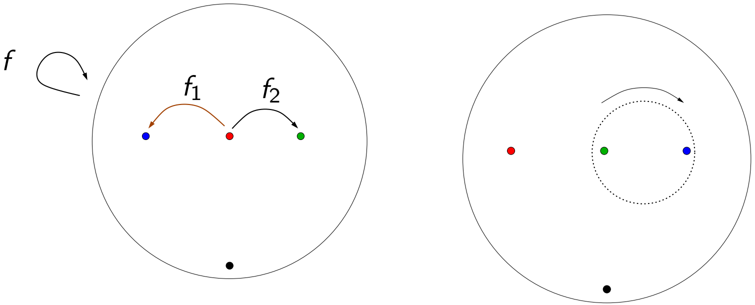

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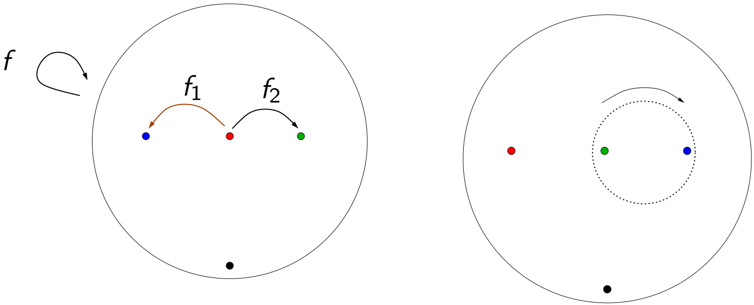

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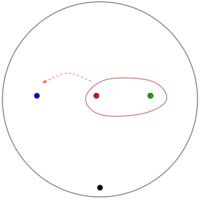

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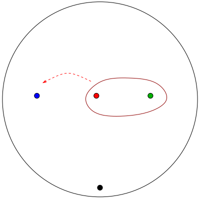

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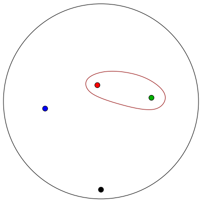

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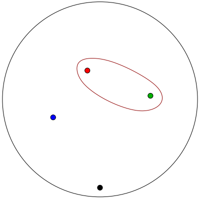

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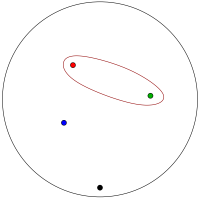

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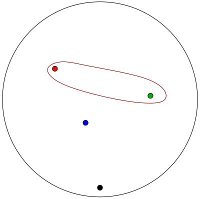

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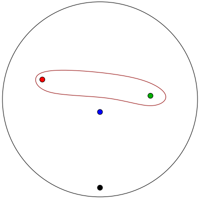

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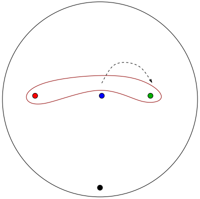

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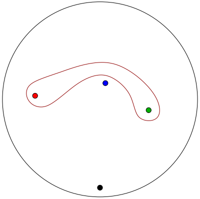

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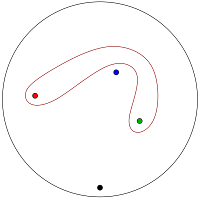

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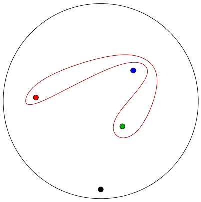

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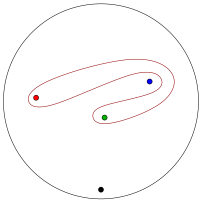

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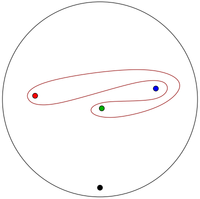

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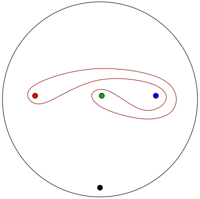

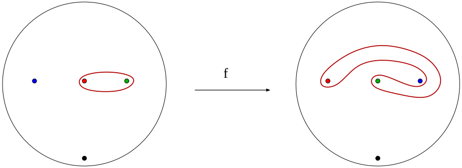

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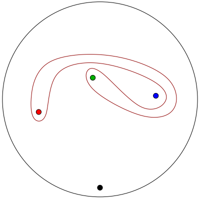

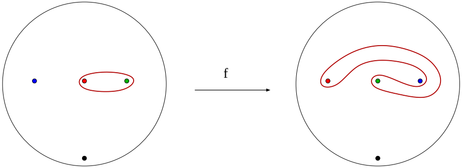

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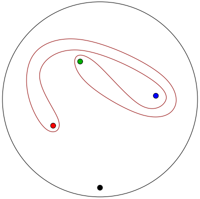

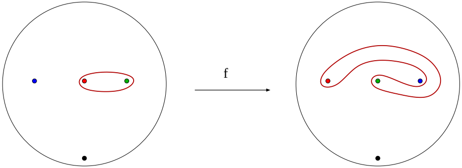

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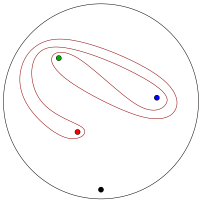

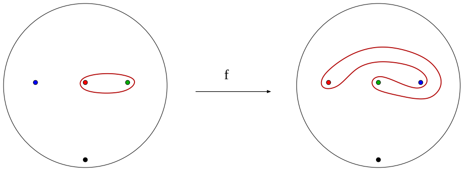

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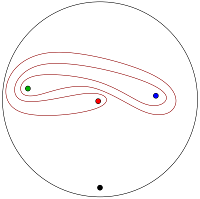

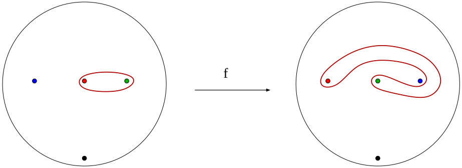

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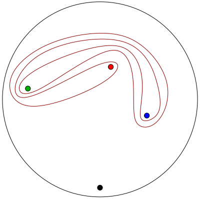

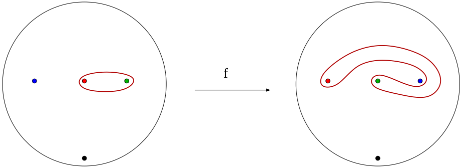

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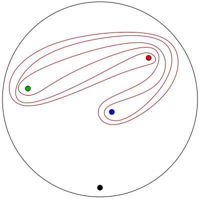

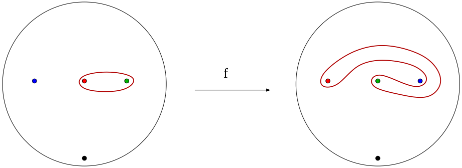

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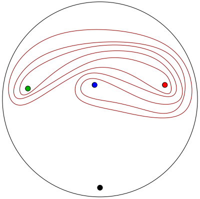

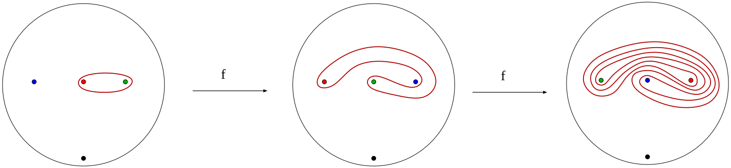

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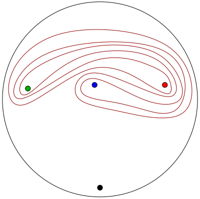

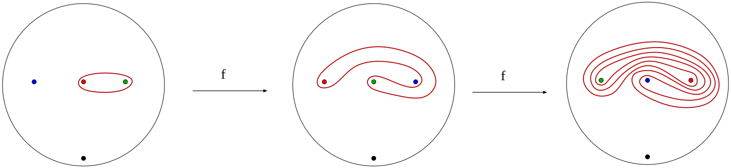

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Fact

$\{\log \lambda(f) | f \in Mod(S_{g,n})$ pseudo-Anosov$\}$ is discrete.
(Arnoux-Yoccoz '81, Ivanov '88)

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  Combining with Penner's lower bound, $\Rightarrow \frac{\log 2}{4n-12} \leq l_{0,n} < \frac{2\log(2+\sqrt{3})}{n-3}$, for $n \geq 4$.

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Question: (Penner)

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How does $l_{g,n}$ behave?

(g,n)-rays	**The asymptotic behavior of $l_{g,n}$**
$g=0$	$1/n \approx 1/\chi(S_{g,n})$
$g=1$ and n even	$1/n \approx 1/\chi(S_{g,n})$
$g=$constant≥ 2	$\log n/n \approx \log \chi(S_{g,n})/\chi(S_{g,n})$
$n=0,1,2,3or4$	$1/g \approx 1/\chi(S_{g,n})$
$n=g,g+1org+2$	$1/g \approx 1/\chi(S_{g,n})$
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Question

What are asymptotic behaviors of $l_{g,n}$ along different (g,n)-rays in the (g,n) plane?

Sketch of the proof:

Lower Bound: $\frac{\log n}{c_g n} < l_{g,n}$

For any pseudo-Anosov $f \in Mod(S_{g,n})$, we consider the forgetting mapping class $\widehat{f} \in Mod(S_{g,0})$.

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1. a pseudo-Anosov homeomorphism on a connected subsurface.
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$$\log \lambda(\widehat{f}^{\alpha}) \geq \frac{\log 2}{12g_0 - 12 + 4n_0} \geq \frac{\log 2}{12g - 12}$$

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$$\Rightarrow \log \lambda(f) \geq \log \lambda(\widehat{f}) > \frac{\log 2}{\alpha(12g - 12)}.$$

This lower bound only depends on g.

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$\Rightarrow \log \lambda(f) \geq \frac{\log k}{2\alpha k} \geq \frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)}$.