MODEL THEORY OF $\mathbb{R}$-TREES
RESEARCH STATEMENT

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1. Introduction

Mathematical disciplines such as algebraic geometry and number theory benefit from the application of classical first-order logic and model theory. However, classical first-order logic does not deal effectively with mathematical structures based on metric spaces. Continuous logic as developed in [1] and [2] is an extension of first-order logic that expands the scope of model-theoretic tools to include structures from analysis and geometry in a natural way. It grew out of several earlier approaches to this challenge, including Henson's positive bounded logic for Banach spaces ([11], [12]), Ben Yaacov's metric compact abstract theories ([4]), and $[0,1]$-valued logics studied by Chang and Keisler ([5]).

The perspective of mathematical logic has already been useful to geometric group theorists. For example, in [8] van den Dries and Wilkie constructed the asymptotic cone of a finitely generated group. Their work helped make explicit, via ultraproduct constructions, ideas about the convergence of metric spaces that were needed in geometric group theory. Ultraproduct constructions are a core notion in continuous logic.

My current research concerns the model theory of $\mathbb{R}$-trees, a class of metric spaces of central importance in geometric group theory. Every asymptotic cone of a word hyperbolic group is an $\mathbb{R}$-tree, and group actions on $\mathbb{R}$-trees are widely used tools. In my work so far, I have axiomatized the class of $\mathbb{R}$-trees in an appropriate continuous signature, found a model companion for that theory, and explored various properties of this model companion (Section 4). I have also examined what happens when an isometry is added to the structure (Section 5), and begun to explore model-theoretic aspects of group actions on $\mathbb{R}$-trees (Section 6).

2. Continuous logic background

Like a classical first-order signature, a **bounded continuous signature** L consists of symbols for predicates, constants, and functions. In the continuous setting, each function symbol and predicate symbol comes with a specified modulus of uniform continuity, and arity. Instead of $=$, we use a symbol d, which will be interpreted by a complete metric. For the metric symbol and each predicate symbol we specify a bounded interval, conventionally $[0,1]$, in which the interpretation must take its values. Terms and formulas are built inductively in the normal way, and atomic formulas are predicate symbols or metric symbols applied to terms. The connectives consist of the set of continuous functions from $[0,1]^n \to [0,1]$, and the quantifiers are $\sup_x$ and $\inf_x$. In practice, using only the connectives $x \dot{-} y := \max\{x-y, 0\}, \frac{x}{2}, 0, 1, \max$, and min is sufficient.

An **L-structure** $\mathcal{M}$ consists of a complete, bounded metric space (M,d), along with interpretations for each symbol in the signatures L. Note that $\mathcal{M}$ denotes the structure, while M denotes

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the underlying space. An n-ary predicate symbol is interpreted by a function from M^n into the specified interval, an n-ary function symbol is interpreted by a function from M^n to M, and these must satisfy the given moduli of uniform continuity. A constant is interpreted by an element of M. As a result, the interpretation of a formula $\varphi(x_1, ..., x_n)$ in a structure $\mathcal{M}$ is a uniformly continuous function from M^n to the interval $[0, 1]$.

For L-formulas φ and ψ, we call a statement of the form $\varphi = \psi$ an **L-condition**. It is these L-conditions that are either true or false in a given structure. The interpretation of an L-sentence σ in a given structure will be a constant in $[0, 1]$. A **closed** L-condition is one of the form $\sigma = r$, where σ is an L-sentence and $r \in [0, 1]$. A continuous **L-theory** is a collection of closed L-conditions. For an arbitrary L-theory T, we say an L-structure $\mathcal{M}$ is a **model** of T if every closed L-condition in T is true in $\mathcal{M}$. Note that any structure of first-order logic is made into a structure of continuous logic via the use of the discrete, $\{0, 1\}$-valued metric. For more details about continuous logic, see [1] and [2].

Bounded continuous logic deals nicely with metric structures like probability algebras, which are naturally bounded, and Banach spaces we may scale down to the unit ball. In practice, we often want to talk about unbounded metric spaces. One approach is to use a many-sorted signature, with a basepoint p, a sort for each closed ball of integer radius centered at p, and inclusion maps to maintain the relationship between the sorts. It is this type of setting which I use to discuss $\mathbb{R}$-trees in my thesis. Another approach is to use a gauged signature as laid out in [3]. It is convenient to take this perspective when we add an isometry to the signature, but the background details are omitted from this statement.

In the continuous logic setting, as opposed to the classical setting, there is an element of approximation. For example, the $\inf_x$ quantifier does not act exactly like the existential quantifier $\exists x$. The condition $\inf_x \varphi(x) = 0$ guarantees the existence of a sequence of elements making $\varphi(x)$ arbitrarily small, but it does not guarantee the existence of an element satisfying the infimum. The $\sup_x$ quantifier does act like the universal quantifier $\forall x$, since $\sup_x \varphi(x)^{\mathcal{M}} = 0$ is true if and only if $\varphi(a) = 0$ for all $a \in M$.

Model theoretic concepts in first-order logic have natural extensions to continuous logic. Some of these are outlined below, for details, see [1] and [2]. A continuous theory T has **quantifier elimination (Q.E.)** if every formula can be uniformly approximated by quantifier free formulas. Given an cardinal κ, we say a metric theory T is **κ-categorical** if any two models of density character κ are isomorphic. A **complete** continuous theory T is one where all the models of T are elementarily equivalent, that is, they satisfy exactly the same L-conditions. Let T be an L-theory for a given continuous signature L. Let $\mathcal{M} \models T$. We say $\mathcal{M}$ is an **existentially closed (e.c.)** model of T if, for any $\mathcal{N} \models T$ that is an extension of $\mathcal{M}$, any inf-formula $\inf_{x_1} ... \inf_{x_n} \varphi(x_1, ..., x_n, y_1, ..., y_m)$ (where $\varphi(x_1, ..., x_n, y_1, ..., y_m)$ is quantifier free), and any $a_1, ..., a_m \in M$ we have:

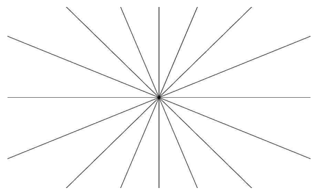
$$\inf_{x_1} ... \inf_{x_n} \varphi(x_1, ..., x_n, a_1, ..., a_m)^{\mathcal{N}} = \inf_{x_1} ... \inf_{x_n} \varphi(x_1, ..., x_n, a_1, ..., a_m)^{\mathcal{M}}.$$

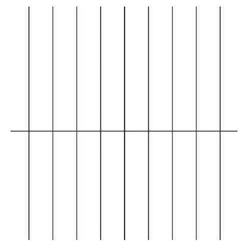
Let T be a theory in a continuous signature L. Let $S \supseteq T$ be another L-theory. We say S is a **model companion** of T if every model of S is an e.c. model of T. Model theorists will recognize the previous definition as one of several equivalent characterizations of a model companion.

3. $\mathbb{R}$-tree background

An **$\mathbb{R}$-tree** is a metric space M such that between any two points in M there is a unique arc and that arc is a geodesic segment. For more information see [6] or [7]. In an $\mathbb{R}$-tree we will use $[a, b]$

to denote the unique geodesic segment between a and b. For example, $\mathbb{R}^2$ with the Paris metric, pictured below at left, is an $\mathbb{R}$-tree.





Above at right is a picture of $\mathbb{R}^2$ with the metric

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2 \\ |y_1| + |y_2| + |x_1 - x_2| & \text{if } x_1 \neq x_2 \end{cases}$$

which is an $\mathbb{R}$-tree. Any simplicial tree can be made into an $\mathbb{R}$-tree by making each edge an isometric copy of $[0, 1]$. With this perspective, the Cayley graph of a free group is an $\mathbb{R}$-tree.

Let M be an $\mathbb{R}$-tree and $a \in M$. Call the connected components of $M \setminus \{a\}$ **branches** at a. The **height** of a branch B at a is $\sup\{d(a, x) | x \in B\}$ if that supremum exists, and is ∞ otherwise. For a metric space M and $x, y, w \in M$, define the **Gromov product**

$$(x \cdot y)_w = \frac{1}{2}[d(x, w) + d(y, w) - d(x, y)].$$

A metric space M is **0-hyperbolic** if for all $w, x, y, z \in M$,

$$\min\{(x \cdot z)_w, (y \cdot z)_w\} \leq (x \cdot y)_w.$$

If M is geodesic, this is equivalent to the condition: given $a, b, c \in M$ and any geodesic segments $[a, b], [b, c]$, and $[c, a]$, the segment $[a, b] \subseteq ([b, c] \cup [c, a])$. This means that any geodesic triangle is a tripod (shaped like a "Y"). Any $\mathbb{R}$-tree is 0-hyperbolic. Moreover, any 0-hyperbolic metric space embeds isometrically in an $\mathbb{R}$-tree. This characterization is helpful when finding axioms for the class of $\mathbb{R}$-trees.

Let $h > 0$. An $\mathbb{R}$-tree M is **h-richly branching** if

$$B_h := \{b \in M | \text{ at b there are at least 3 branches of height } \geq h\}$$

is dense in M. If M is h-richly branching for some $h > 0$, we say it is **richly branching**. It is straightforward to show that if M is h-richly branching for some $h > 0$, then it is h-richly branching for every $h > 0$.

Let μ be a cardinal. An $\mathbb{R}$-tree M is called **μ-universal** if any $\mathbb{R}$-tree with $\leq \mu$ branches at each point can be isometrically embedded in M ([9] and [10]). In [10], the authors construct a μ-universal $\mathbb{R}$-tree A_μ for any cardinal $\mu \geq 2$. Their space A_μ is a complete $\mathbb{R}$-tree that is homogeneous, with μ branches at every point. They show that any complete $\mathbb{R}$-tree with μ-branches at each point is isometric to A_μ. For $\mu \geq 3$, clearly A_μ is an example of a richly branching $\mathbb{R}$-tree. Every asymptotic cone of an arbitrary non-elementary hyperbolic group is isometric to the 2^ω-universal $\mathbb{R}$-tree A_{2^ω} ([9].)

4. Selected $\mathbb{R}$-tree results

This section presents many of my results concerning the continuous theory of $\mathbb{R}$-trees. Let L be a suitable continuous signature containing a constant symbol p. For the sake of brevity, the details of the setting and of the signature are omitted.

4.1. **Theorem.** *The class of complete, pointed* $\mathbb{R}$*-trees is axiomatizable in* L*.*

The theory of this class, $\mathbb{R}$T, consists of L-conditions equivalent to the following:

- Approximate midpoint property (AMP):
  $\sup_x \sup_y \inf_z \max \{|d(x,z) - \frac{d(x,y)}{2}|, |d(y,z) - \frac{d(x,y)}{2}|\} = 0$
- 0-hyperbolic property:
  $\sup_x \sup_y \sup_z \sup_w \left(\min\{(x \cdot z)_w, (y \cdot z)_w\} \dot{-} (x \cdot y)_w\right) = 0$

4.2. **Theorem.** *The* L*-theory* $\mathbb{R}T$ *has amalgamation over substructures.*

To find a model companion to $\mathbb{R}$T, it is enough to axiomatize the class of e.c. models of $\mathbb{R}$T. In order for $\mathcal{M} \models \mathbb{R}$T to be an e.c. model, there must be enough branching to allow the (approximate) embedding of any finitely generated $\mathbb{R}$-tree anywhere in M. In my thesis I establish the following results.

4.3. **Theorem.** *If* $\mathcal{M} \models \mathbb{R}T$*,* $\mathcal{M}$ *is e.c. if and only if the underlying* $\mathbb{R}$*-tree* M *is richly branching.*

4.4. **Theorem.** *The class of pointed, richly branching* $\mathbb{R}$*-trees is axiomatizable in* L*.*

Denote the theory of this class by rb$\mathbb{R}$T.

4.5. **Corollary.** *The theory rb*$\mathbb{R}T$ *is the model companion of* $\mathbb{R}T$*.*

When studying a particular theory, there are a number of questions commonly asked by model theorists. For example, is the theory complete? Does it have Q.E.? In what cardinals is the theory categorical? In what cardinals is it stable? I have established the following properties of rb$\mathbb{R}$T.

4.6. **Theorem.**

(1) The L*-theory rb*$\mathbb{R}T$ *has Q.E. and is complete.*
(2) The L*-theory rb*$\mathbb{R}T$ *is not categorical in any cardinal.*
(3) A model of rb$\mathbb{R}T$ *is* κ*-saturated if and only if the underlying space has at least* κ *branches at every point.*
(4) When κ *is an infinite cardinal, rb*$\mathbb{R}T$ *is* κ*-stable if and only if* κ *satisfies* $\kappa^\omega = \kappa$*.*

I have also characterized the independence relation for rb$\mathbb{R}$T. Let κ be a cardinal greater than than card(L). Let U be a κ-universal domain for rb$\mathbb{R}$T.

4.7. **Definition.** Let A, B and C be small subsets of U. Say A is *-independent from B over C, denoted $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$, if and only if $\operatorname{dist}(a, \overline{E_{C \cup B}}) = \operatorname{dist}(a, \overline{E_C})$ for all $a \in A$.

4.8. **Theorem.** *The* **-independence relation is the model-theoretic (non-dividing) independence relation for rb*$\mathbb{R}T$*.*

5. Generic isometries of ℝ-trees

Once the theory of a given class of structures has been studied, a common next step for model theorists is to add an automorphism or an isometry to the structure. In the case of ℝ-trees, studying structures with one isometry is also a natural intermediate step towards studying the model theory of group actions by isometry on ℝ-trees. This section summarizes results from my thesis concerning the model theory of ℝ-trees with an isometry.

If an isometry f of an ℝ-tree M has a fixed point it is called **elliptic**, otherwise it is **hyperbolic**. The quantity $||f|| := \inf_{x\in M} d(x, f(x))$ is called the **translation distance** of f. If $||f|| = 0$, then f is elliptic; if not, then f is hyperbolic and acts as a translation along a copy of $\mathbb{R}$ in M, with points on this axis being moved exactly by distance $||f||$.

For $s > 0$, let L_s be a suitable continuous signature with a constant p and a function f, in which it is specified that $d(p, f(p)) \leq s$. For $0 < r \leq s$, let $K_{r,s}$ be the class of L_s-structures (M, d, p, f) where (M, d, p) is a complete, pointed ℝ-tree and f is a hyperbolic isometry of M such that $\inf_x d(x, f(x)) \geq r$. For convenience, we require p to be on the axis of f. Let $\mathrm{H\mathbb{R}T}_{r,s}$ be the L_s-theory of $K_{r,s}$.

5.1. **Theorem.**

(1) For $0 < r \leq s$, the L_s-theory $H\mathbb{R}T_{r,s}$ is axiomatizable.
(2) For $0 < r \leq s$, the L_s theory $H\mathbb{R}T_{r,s}$ has the amalgamation property over substructures.
(3) A model $\mathcal{M} = (M, d, p, f)$ of $H\mathbb{R}T_{r,s}$ is an e.c. model of $H\mathbb{R}T_{r,s}$ if and only if M is a richly branching ℝ-tree.
(4) For $0 < r \leq s$, $H\mathbb{R}T_{r,s}$ has a model companion.

Call the model companion theory $\mathrm{rbH\mathbb{R}T}_{r,s}$.

5.2. **Theorem.**

(1) Each theory $rbH\mathbb{R}T_{r,s}$ has Q.E..
(2) If we add an axiom specifying $||f||$ exactly, we get a completion of $rbH\mathbb{R}T_{r,s}$.
(3) Each theory $rbH\mathbb{R}T_{r,s}$ is κ-stable if and only if $\kappa^\omega = \kappa$.

For the elliptic case, let L_0 be L_s as defined above for $s = 0$. Let K_0 be the class of L_0-structures (M, d, p, f) where M is an ℝ-tree, f is an elliptic isometry of M and p is a fixed point of f in M. Let $\mathrm{E\mathbb{R}T}$ be the L_0-theory of K_0.

5.3. **Theorem.**

(1) The L_0-theory $E\mathbb{R}T$ is axiomatizable.
(2) The L_0-theory $E\mathbb{R}T$ has the amalgamation property over substructures.
(3) The L_0-theory $E\mathbb{R}T$ has a model companion.

Call the model companion theory $\mathrm{rbE\mathbb{R}T}$. This model companion is not as straightforward to axiomatize as in the hyperbolic case where all we need to guarantee is that the underlying ℝ-tree is richly branching (see Theorem 5.1 (3)). In the elliptic case, when finding the axioms which capture e.c. models of $\mathrm{E\mathbb{R}T}$, there must be axioms which ensure that the behavior of the isometry is rich enough. The axioms are natural, but complex, so I have omitted the specifics.

5.4. **Theorem.**

(1) The L_0-theory $rbE\mathbb{R}T$ has Q.E. and is complete.
(2) The L_0-theory $rbE\mathbb{R}T$ is κ-stable if and only if $\kappa^\omega = \kappa$.

The independence relations for rbE$\mathbb{R}T$ and rbH$\mathbb{R}\mathrm{T}_{r,s}$ are derived from the independence relation for rb$\mathbb{R}$T.

6. Group actions on $\mathbb{R}$-trees and future work

I have begun to investigate the model theory of isometric actions of arbitrary finitely generated groups on $\mathbb{R}$-trees. I have found model companions for two extreme types of group actions on $\mathbb{R}$-trees in suitable continuous signatures. One extreme is when all of the group elements correspond to hyperbolic isometries with a uniform lower bound on their translation distances. In this case a theorem in [13] implies that the group involved must be a free group. The other extreme is when all the group elements correspond to elliptic isometries. This is what geometric group theorists call a "trivial" action. The model companions in these cases are analogous to those in the case of a single isometry.

Studying these extreme cases is my first step towards studying group actions on $\mathbb{R}$-trees in general. What can be said about actions of finitely generated groups that have a mixture of elliptic and hyperbolic isometries? For example, if we fix a group G and specify data such as a translation distance $||g||$ for each $g \in G$, is there a model companion? If so, what are the axioms and what properties does it have? What is definable in such a structure? Also, very generally, what are the connections to geometric group theory?

Perturbations as in [3] are another immediate area of exploration. What classes of maps between $\mathbb{R}$-trees are valid perturbations? What properties does rb$\mathbb{R}$T have up to perturbation? Any perturbation between $\mathbb{R}$-trees will be a homeomorphism, so it is already clear that rb$\mathbb{R}$T will not be categorical up to perturbation. But, is rb$\mathbb{R}$T ω-stable or superstable up to perturbation? I also plan to study perturbations on $\mathbb{R}$-trees with an isometry and on $\mathbb{R}$-trees with a group action.

The original inspiration for my thesis project was the investigation of the continuous model theory of asymptotic cones of finitely generated groups, which remains a rich area of potential applications for continuous logic.

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