Math 241 Discussion Section-Work sheet

12/9/2008 Review-Final

Part I (Chapter 14.6-14.8)

1. Find the center of the mass of the portion of $\phi = \pi/4$ above the region bounded by $y = x^2$ and $y = 1$, and the density $\delta(x, y, z) = z$. (**Only need to set up**)

S is a cone $z = r = \sqrt{x^2+y^2} = f(x,y)$

so $dS = \| \langle f_x, f_y, -1 \rangle \| \, dA = \sqrt{2}\, dA$

$m = \iint_S \delta \, dS = \iint_S z \, dS = \int_{x=-1}^{1} \int_{y=x^2}^{1} \sqrt{x^2+y^2} \sqrt{2}\, dy\, dx$

$\bar{x} = 0$ by symmetry

$\bar{y} = \frac{1}{m} \iint_S \delta y \, dS = \frac{1}{m} \int_{x=-1}^{1} \int_{y=x^2}^{1} \sqrt{x^2+y^2}\, y \sqrt{2}\, dy\, dx$

$\bar{z} = \frac{1}{m} \iint_S \delta z \, dS = \frac{1}{m} \int_{x=-1}^{1} \int_{y=x^2}^{1} \sqrt{x^2+y^2} \sqrt{x^2+y^2} \sqrt{2}\, dy\, dx$

center $= (\bar{x}, \bar{y}, \bar{z})$

2. Find the line integral of $\vec{F} = \langle e^{x^2}, \sin y, z^8 \rangle$ over the unit circle on xy-plane with counterclockwise orientation viewing from the positive z-axis.

$\oint_C \vec{F} \cdot d\vec{r} = ?$ You can see it's hard to intergrate directly.

What we have ?

<I> $\vec{F}$ is conservative $\Leftrightarrow \vec{\nabla} \times \vec{F} = \vec{0} \Leftrightarrow \oint_C \vec{F} \cdot d\vec{r} = 0 \ \forall C \Leftrightarrow$ others

<II> Stoke's Thm $\int_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot dS, \ \partial S = C$

(From here you can see why <I> is true)

Check $\vec{\nabla} \times \vec{F} = \vec{0}$ ✓

so $\boxed{\oint_C \vec{F} \cdot d\vec{r} = 0}$

Remark: $\vec{\nabla} \times \vec{F} \neq \vec{0}$, we can use Stoke's Thm.
make sure you have positive orientation
and continuous 1st partial derttive.
Similar idea as Divergence Thm, if $\vec{F}$ is too messy, we hope it will "clean up" by $\vec{\nabla}$

3. Find the flux of $\vec{F} = \langle e^y + x^2, \ln xz, 1\rangle$ over the surface S, where S is the portion of the cone $z = r$ below $z = 2$. ($\vec{n}$ is the usual outward orientation.)

$S: \ z = \sqrt{x^2+y^2}$ $\qquad \vec{n} = \dfrac{\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle}{\left\| \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle \right\|}$ (outward)

$dS = \left\| \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle \right\| dA$

$\iint_S \vec{F}\cdot\vec{n}\, dS$ seems hard.

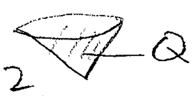
As the Remark in #2. we want to use Div. Thm

since $\vec{\nabla}\cdot\vec{F} = 2x$ & much simpler !!

Note $\vec{F}$ has continuous 1st partial derivative everywhere

I can choose any Q I want

One trivial one will be



Q = solid bdd between $z=r$ & $z=2$

① so $\iiint_Q \vec{\nabla}\cdot\vec{F}\, dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=r}^{2} \underbrace{2r\cos\theta}_{2x} \underbrace{r\,dz\,dr\,d\theta}_{dV} = 0$

② $\partial Q = S \cup S'$ $\quad S': z=2$ inside $z=r$ $\Rightarrow \vec{n} = \langle 0,0,1\rangle$

$\iint_{S'} \vec{F}\cdot\vec{n}\, dS = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} 1\cdot\|\langle 0,0,-1\rangle\|\, r\,dr\,d\theta = 4\pi$

Div. Thm $\underbrace{\iiint_Q \vec{\nabla}\cdot\vec{F}\, dV}_{= 0} = \iint_{\partial Q} \vec{F}\cdot\vec{n}\, dS = \iint_S \vec{F}\cdot\vec{n}\, dS + \underbrace{\iint_{S'} \vec{F}\cdot\vec{n}\, dS}_{= 4\pi}$

so $\iint_S \vec{F}\cdot\vec{n}\, dS = \boxed{-4\pi}$

$$\vec{F} = \frac{\vec{r}}{r^3} = \frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$$

4. Compute the flux of the inverse field of constant 1 over a surface S. ($\vec{n}$ is the usual outward orientation.)

(a) S is the portion of $z = 2 - x^2 - y^2$ above $z = 1$.

If you compute $\iint_S \vec{F}\cdot\vec{n}\,dS$ directly, you will get

$$\iint_S \vec{F}\cdot\vec{n}\,dS = 2\pi\int_{r=0}^{1} \frac{2r+r^3}{(-3r^2+r^4+4)^{3/2}}\,dr \quad \leftarrow \text{hard to compute!}$$

Use Divergence Thm.

Let S': $z=1$ bdd inside $z=2-x^2-y^2$, then Q is the solid between $z=2-x^2-y^2$ & $z=1$

$\partial Q = S \cup S'$

and $\vec{F}$ has conti. 1st partial derivative.

So $\iiint_Q \vec{\nabla}\cdot\vec{F}\,dS = \iint_S \vec{F}\cdot\vec{n}\,dS + \iint_{S'} \vec{F}\cdot\vec{n}\,dS$

Check $\vec{\nabla}\cdot\vec{F} = 0$

Check $\iint_{S'} \vec{F}\cdot\vec{n}\,dS = \iint_R \frac{-1}{(x^2+y^2+1)^{3/2}}\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \frac{-1}{(r^2+1)^{3/2}}\,r\,dr\,d\theta = \frac{2\pi}{\sqrt{2}} - 2\pi$

So $\iint_S \vec{F}\cdot\vec{n}\,dS = -\left(\frac{2\pi}{\sqrt{2}} - 2\pi\right)$

$= \boxed{2\pi\left(1-\frac{1}{\sqrt{2}}\right)}$

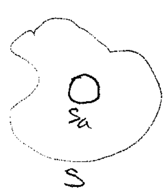
(b) S is any closed surface which encloses the origin.

Same as the worksheet on Dec. 4th

Choose S_a = sphere with radius a center at the origin

to be another boundary.

Since

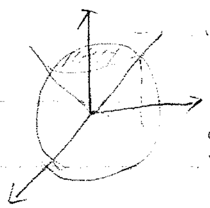


you want to adjust a to make sure your S_a is enclosed by S.

But you will see the value a doesn't affect the solution.

Part II | $S: z = f(x,y)$

#1. Find the area of the surface S where S lies on the sphere of radius 2 and bounded above $z = r$



(a) use surface integral

Note: this is the same as the formula in §13.4

$S: z = \sqrt{4-x^2-y^2}$ over $R: x^2+y^2 \le (2\cos\frac{\pi}{4})^2 = 2$

$$A(S) = \iint_S dS = \iint_R \sqrt{f_x^2+f_y^2+1}\, dA = \iint \| \langle f_x, f_y, -1 \rangle \| \, dA$$

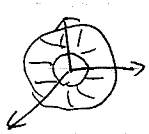
$$= \iint_R \sqrt{\frac{x^2}{4-x^2-y^2} + \frac{y^2}{4-x^2-y^2} + 1}\; dA$$

$$= \iint_R \frac{2}{\sqrt{4-x^2-y^2}}\, dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{2}} \frac{2}{\sqrt{4-r^2}}\, r\,dr\,d\theta = \boxed{4\pi(2-\sqrt{2})}$$

(b) Use (ϕ, θ)

$$A(S) = \iint_S dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4} 2^2 \sin\phi \, d\phi\, d\theta = 8\pi\left(\frac{2-\sqrt{2}}{2}\right) = \boxed{4\pi(2-\sqrt{2})}$$

$dS = \rho^2 \sin\phi\, d\phi\, d\theta$



(a) $$\iiint_Q dV = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=1}^{2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 2\pi \left.\frac{\rho^3}{3}\right|_{\rho=1}^{2} \left. -\cos\phi \right|_{\phi=0}^{\pi}$$

$$= \frac{2}{3}\pi \times 7 \times (+1+1) = \boxed{\frac{28\pi}{3}}$$

(b) $$\iiint_Q dV = \iiint_Q \vec{\nabla}\cdot\vec{F}\, dV = \iint_{\partial Q} \vec{F}\cdot\vec{n}\, dS$$

$\vec{F} = \langle 0, 0, z \rangle \qquad \vec{n} = \pm \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$

$$\iiint_Q dV = \iint_{S_1} \vec{F}\cdot\vec{n}\, dS + \iint_{S_2} \vec{F}\cdot\vec{n}\, dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} -\cos^2\phi \sin\phi\, d\phi\, d\theta + \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} 2\cos^2\phi\, 2^2 \sin\phi\, d\phi\, d\theta$$

S_1: $\vec{n}$ pointed the center $\qquad$ S_2: $\vec{n}$ point outward

$$= 7 \times 2\pi \times \int_{\phi=0}^{\pi} \cos^2\phi \sin\phi\, d\phi = 14\pi \int_1^{-1} -u^2\, du = 14 \left.\frac{-u^3}{3}\right|_1^{-1} = \boxed{\frac{28}{3}\pi}$$

$u = \cos\phi : 1 \to -1$

$du = -\sin\phi\, d\phi$

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Part IV

1. Determine wheather the following statements are true or false, and need to show the work.

   F (a) If $\vec{r} = \langle \cos t, \sin t \rangle$, $\vec{r}$ and $\vec{r}'$ are parallel.

   $\|\vec{r}\| = \sqrt{\cos^2 t + \sin^2 t} = 1$ is a constant $\Leftrightarrow$ $\vec{r}$ & $\vec{r}'$ are orthogonal

   P.871 Thm 2.4

   T (b) $\|\vec{T} \times \vec{a}\| = a_N$. (Review a_N and a_T.) P.901 §11.2–11.5

   $\|\vec{T} \times \vec{a}\| = \|a_N \vec{B}\| = |a_N| \, \|\vec{B}\| = |a_N|$; $a_N \geq 0$, $\| a_N$

   $\vec{T}, \vec{N}, \vec{B}$ are unit vectors

   $\vec{a} = a_T \vec{T} + a_N \vec{N}$

   $\vec{T} \times \vec{a} = \vec{T} \times (a_T \vec{T}) + \vec{T} \times (a_N \vec{N})$

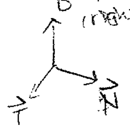
   $= a_T (\vec{T} \times \vec{T}) + a_N (\vec{T} \times \vec{N}) = a_N \vec{B}$

   since $\vec{T} \times \vec{T} = 0 \Leftrightarrow \vec{T} \parallel \vec{T}$

   since $\vec{T} \times \vec{N} = \vec{B}$ By definition

2. Determine which one of the following has value 0, and need to show the work.

   $\vec{B} = \vec{T} \times \vec{N}$ (right-hand rule)

   (a) $\frac{d}{ds}(\vec{T} \bullet \vec{B}) = \frac{d}{ds}(0) = 0$ since $\vec{T}, \vec{N}, \vec{B}$ are mutually orthogonal

   (b) $\vec{T} \bullet (\vec{N} \times \vec{B}) = \vec{T} \bullet \vec{T} = \|\vec{T}\|^2 \cos 0 = \|\vec{T}\|^2 = 1$ $\because \vec{T}$ is a unit vector

   (c) $\vec{T} \bullet (\frac{d\vec{T}}{ds}) = 0$ since $\|\vec{T}\| = 1 \Leftrightarrow \vec{T}$ and $\frac{d\vec{T}}{ds}$ are orthogonal (same as #1(a))

(b)

3. Let $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$ be, respectively, position, velocity, and acceleration at time t. Computing the following and simplifying as much as possible. Thm 2.3 P.868 $\vec{r}' = \vec{v}$, $\vec{v}' = \vec{a}$

   (a) $\frac{d}{dt}(\vec{r} \times m\vec{v}) = \vec{r}' \times (m\vec{v}) + \vec{r} \times (m\vec{v}')$

   $= m(\underbrace{\vec{v} \times \vec{v}}_{0,\ \vec{v} \parallel \vec{v}}) + m(\vec{r} \times \vec{a}) = \boxed{m(\vec{r} \times \vec{a})}$

   (a) Compute $\frac{d}{dt}(\vec{r}(t) \times m\vec{v}(t))$, where m is the mass.

   (b) Let $V(t)$ be the scalar triple product of $\vec{r}(t)$, $\vec{v}(t)$, and $\vec{a}(t)$. Compute $V'(t)$. in Exam I

   $V = \vec{r} \cdot (\vec{v} \times \vec{a}) \Rightarrow V' = \vec{r} \cdot (\vec{v} \times \vec{a}')$

4. Label each expression as a scalar quantity, a vector quantity or undefined, if f is a scalar function and $\vec{F}$ is a vector field. Indicate which one is 0 and $\vec{0}$.

   (a) $\text{div}(\vec{\nabla} f) = \vec{\nabla} \cdot (\vec{\nabla} f) = \vec{\nabla} \cdot (\text{vector}) = \text{scalar}$

   (b) $\vec{\nabla}(\vec{\nabla} \bullet \vec{F}) = \vec{\nabla}(\text{scalar}) = \text{vector}$

   (c) $\vec{\nabla}(\vec{\nabla} f) = \vec{\nabla}(\text{vector})$ undefined

   (d) $\text{curl}(\vec{\nabla} f) = \vec{\nabla} \times (\vec{\nabla} f) = \underset{\text{vector}}{\vec{0}}$ $\because \vec{\nabla} f$ is a conservative vector field, then Thm 5.1 P.1170

   (e) $\text{div}(\text{curl} \vec{F}) = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = \vec{\nabla} \cdot (\text{vector}) = \text{scalar} = 0$

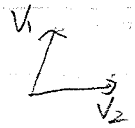
   $\vec{\nabla} \times \vec{F}$ = a vector $\perp$ to $\vec{\nabla}$ & $\vec{F}$

Part: V (Area/ Volume) Extra.

#1 Find the area of the triangle with vertices $(0,0,0)$, $(2, \overset{2}{\_}, -1)$ and $(\overset{2}{\_}, -1, \overset{1}{\_})$

(a) Use cross product.

$\vec{V_1} = \langle 2, 2, -1 \rangle \qquad \vec{V_2} = \langle 2, -1, 1 \rangle$



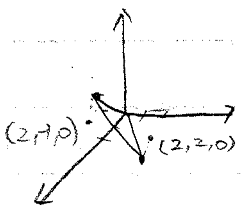
Area of the parallelgrom $= \| \vec{V_1} \times \vec{V_2} \|$

$$\vec{V_1} \times \vec{V_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \langle 2-1, -(2+2), -2-4 \rangle = \langle 1, -4, -6 \rangle$$

$$\| \vec{V_1} \times \vec{V_2} \| = \sqrt{1+16+36} = \sqrt{53}$$

Area of $\triangle$ $= \boxed{\dfrac{\sqrt{53}}{2}}$

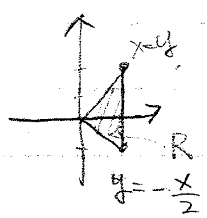
(b) Use double integral



From (a) we know $\langle 1, -4, -6 \rangle$ is a normal vector & $(0,0,0)$ is on S

$\Rightarrow S: 1 \cdot (x-0) + (-4)(y-0) + (-6)(z-0) = 0$

$\Rightarrow x - 4y - 6z = 0 \qquad z = \dfrac{x-4y}{6}$



$$A(S) = \iint_S dS = \iint_R \| \langle \tfrac{1}{6}, -\tfrac{4}{6}, -1 \rangle \| \, dA = \iint_R \frac{\sqrt{53}}{6} \, dA$$

$$= \frac{\sqrt{53}}{6} \int_{x=0}^{2} \int_{y=-\frac{x}{2}}^{x} dy\,dx = \frac{\sqrt{53}}{6} \int_{x=0}^{2} x + \frac{x}{2} \, dx$$

$$= \frac{\sqrt{53}}{6} \cdot \frac{3}{2} \cdot \frac{x^2}{2} \Big|_{x=0}^{2} = \frac{\sqrt{53}}{6} \cdot \frac{3}{4} \cdot 4 = \boxed{\frac{\sqrt{53}}{2}}$$

(c) Use Green Thm.

Part V:

#2. Vol of tetrahedron with vertices (0,0,0), (0,0,2), (2,2,0), (-1,2,0)

(a) cross product

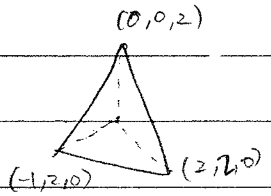
$\vec{V_1} = \langle 0,0,2 \rangle \quad \vec{V_2} = \langle 2,2,0 \rangle \quad \vec{V_3} = \langle -1,2,0 \rangle$

Vol of a parallelepiped $= |\vec{V_1} \cdot (\vec{V_2} \times \vec{V_3})|$

$$\vec{V_1} \cdot (\vec{V_2} \times \vec{V_3}) = \begin{vmatrix} 0 & 0 & 2 \\ 2 & 2 & 0 \\ -1 & 2 & 0 \end{vmatrix} = 0+0+8+4 = 12$$

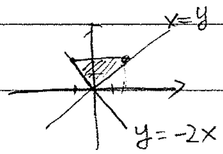
Vol of a tetrahedron $= \frac{1}{6}$ Vol (Parallelepiped) $= \frac{1}{6} \times 12 = \boxed{2}$

(b) Triple integral



$$\iiint_Q dV = \ ?$$

Plane P containing (0,0,2), (-1,2,0), (2,2,0) $\Rightarrow \langle -1,2,-2 \rangle, \langle 2,2,-2 \rangle$



$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -2 \\ 2 & 2 & -2 \end{vmatrix} = \langle -4+4, -(2+4), -2-4 \rangle = \langle 0,-6,-6 \rangle$$

$$0(x-0) - 6(y-0) - 6(z-2) = 0 \Rightarrow y+z=2$$

So
$$\iiint_Q dV = \int_{y=0}^{2} \int_{x=\frac{-y}{2}}^{y} \int_{z=0}^{2-y} dz\,dx\,dy$$

$$= \int_{y=0}^{2} \left(y+\frac{y}{2}\right)(2-y)\,dy = \int_{y=0}^{2} \frac{3}{2}(2y-y^2)\,dy$$

$$= \frac{3y^2}{2} - \frac{y^3}{2}\Big|_{y=0}^{2} = \frac{3 \times 4}{2} - \frac{8}{2} = 6-4 = 2$$