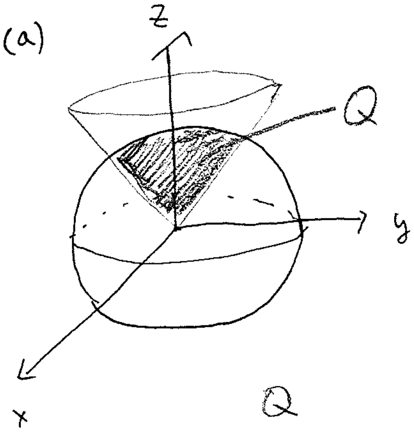
Part II

#1 (same as Part II on 10/28)

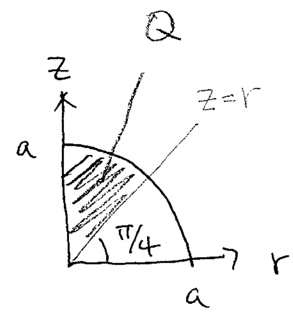
(a)



$0 \le \phi \le \frac{\pi}{4}$

$0 \le \theta \le 2\pi$

$0 \le \rho \le a$



$$\mathrm{Vol}(Q) = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \frac{a^3}{3} (-\cos\phi)\Big|_{\phi=0}^{\pi/4}$$

$$= \boxed{\frac{2\pi a^3}{3}\left(1 - \frac{1}{\sqrt{2}}\right)}$$

(b) $\bar{x} = \bar{y} = 0$ by the symmetry of Q & density $= 1$

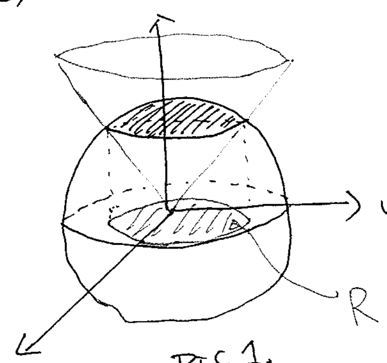
mass $m = \mathrm{vol}(Q)$

$$\bar{z} = \frac{1}{m}\iiint_Q z \cdot \underbrace{1}_{\text{density}} \, dV = \frac{1}{m}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{a} \rho\cos\phi \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

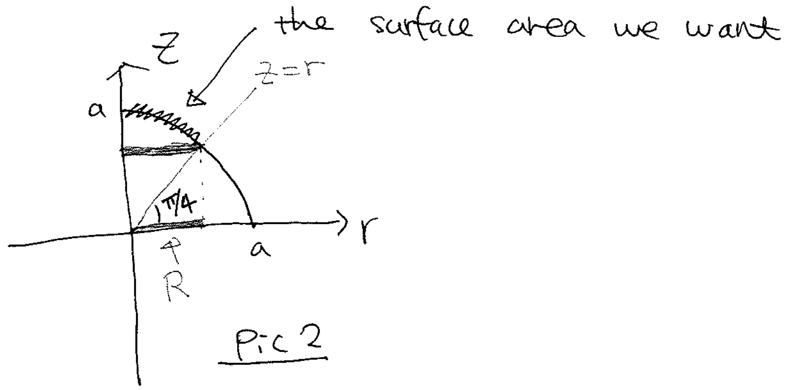
$$= \frac{1}{m} \cdot \frac{1}{8}\pi a^4 = \frac{3a(2+\sqrt{2})}{16}$$

so centroid is $\boxed{\left(0, 0, \frac{3a(2+\sqrt{2})}{16}\right)}$

(c)



Pic 1.



Pic 2

Need: ① surface function $f(x,y)$

② over which region R.

Since we want the part lies on the sphere. $x^2+y^2+z^2=a^2$

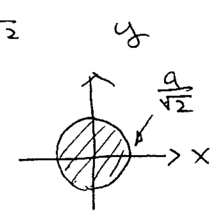
$\Rightarrow$ $\underset{f(x,y)}{\overset{\parallel}{z}}$ is given by the sphere.

$\Rightarrow \quad f(x,y) = \sqrt{a^2-x^2-y^2} \quad$ — ①

We only want the part on the sphere but inside the cone

R must lies on xy-plane. $\Rightarrow z = \sqrt{x^2+y^2} = \sqrt{a^2-x^2-y^2}$

$\Rightarrow x^2+y^2 = \frac{a^2}{2} \quad \Rightarrow R$ is



(or you can see it in Pic 2. radius of $R = a\cos\frac{\pi}{4}$)

So! We can use our formula in §13.4

$$A(S) = \iint_R \sqrt{f_x^2+f_y^2+1}\, dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{\frac{a}{\sqrt{2}}} \frac{a}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta = \boxed{\pi a^2(2-\sqrt{2})}$$

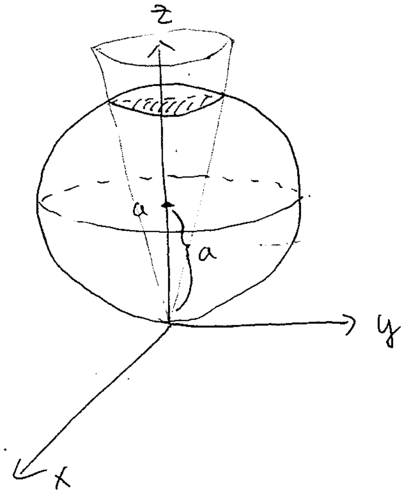
(d) Similar. But surface function is different!

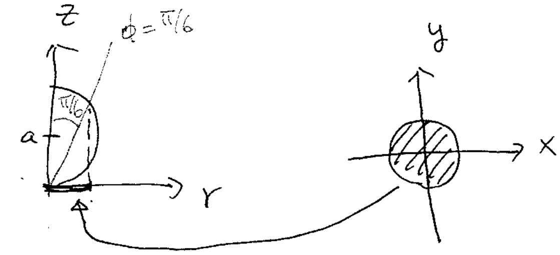
Now we want the part lies on the cone $z=r \Rightarrow f(x,y)=\sqrt{x^2+y^2}$

So $A(S) = \int_{\theta=0}^{2\pi}\int_{r=0}^{\frac{a}{\sqrt{2}}} \sqrt{2}\, r\,dr\,d\theta = \boxed{\frac{\sqrt{2}}{2}\pi a^2}$

Part II

#2. $x^2+y^2+z^2 = 2az \Leftrightarrow x^2+y^2+(z-a)^2 = a^2$





$0 \le \phi \le \pi/6$

so $0 \le \theta \le 2\pi$

Same as #2 in Part I

$x^2+y^2+z^2 = 2az \Rightarrow \rho^2 = 2a\rho\cos\phi \Rightarrow \rho = 2a\cos\phi$, so $0 \le \rho \le 2a\cos\phi$

density

$$m = \iiint_Q \delta \, dV \qquad \text{density} = z = \rho\cos\phi$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/6} \int_{\rho=0}^{2a\cos\phi} \underline{\rho\cos\phi} \cdot \rho^2 \sin^2\phi \, d\rho \, d\phi \, d\theta$$

$$= \boxed{\frac{37}{48}\pi a^4} \leftarrow \text{mass.}$$

Find centroid. Again we have symmetry around z-axis and luckily density $= z$ also have symmetry around z-axis.

$\Rightarrow \bar{x} = \bar{y} = 0$

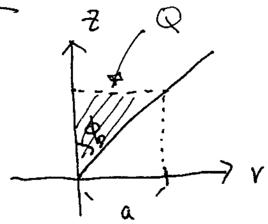
$$z \cdot \delta = z \cdot z = z^2 = (\rho\cos\phi)^2$$

$$\bar{z} = \frac{1}{m} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/6} \int_{\rho=0}^{2a\cos\phi} \underline{(\rho\cos\phi)^2} \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{1}{m} \cdot \frac{35}{32}\pi a^2 = \frac{105a}{74}$$

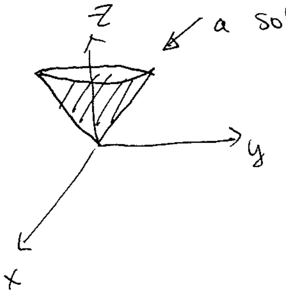
So centroid is $\boxed{\left(0, 0, \frac{105a}{74}\right)}$

Part III

#1.



ϕ_0 is unknown. ⇒ will find out ϕ_0 is given by m & a



a solid cone Q.

It is homogeneous ⇒ density = K ← a constant

(uniform ⇒ density = 1)

$0 \le \theta \le 2\pi$

$0 \le \phi \le \phi_0$

we can see $\rho \sin\phi_0 = a$ ⇒ $0 \le \rho \le \dfrac{a}{\sin\phi_0}$

$$m = \iiint_Q \underbrace{\delta}_{\text{density}} \, dV = \iiint_Q K \, dV = K \iiint_Q dV$$

$$= K \int_0^{2\pi} \int_0^{\phi_0} \int_0^{\frac{a}{\sin\phi_0}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= K \frac{2\pi}{3} \frac{a^3}{(\sin\phi_0)^3} (-\cos\phi) \Big|_{\phi=0}^{\phi_0} = K \frac{2}{3} \frac{\pi a^3}{\sin^3\phi_0} \cdot (-\cos\phi_0 + 1)$$

Note! hard to write ϕ_0 in terms of m, a.

since it's symmetric around z-axis

$$\rho^2 \sin^2\phi (\cos^2\theta + \sin^2\theta) = \rho^2 \sin^2\phi$$

$$I_z = \iiint_Q (x^2+y^2) \delta \, dV = \int_0^{2\pi} \int_0^{\phi_0} \int_0^{\frac{a}{\sin\phi_0}} (\underline{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta}) K \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= K \int_0^{2\pi} \int_0^{\phi_0} \int_0^{\frac{a}{\sin\phi_0}} \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{2\pi K}{5} \frac{a^5}{(\sin\phi_0)^5} \left(\frac{-2\cos\phi_0 - \sin^2\phi_0 \cos\phi_0}{3} \right)$$

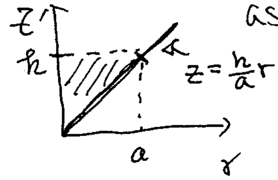
← We don't how how to get rid off ϕ_0

hard to integrate here

So, we should try a different coordinate system.

#1 (Try 2)

How about cylindrical coordinate



assume z goes upto h

$$\left.\begin{aligned} 0 \le z &\le \frac{hr}{a} \\ 0 \le r &\le a \\ 0 \le \theta &\le 2\pi \end{aligned}\right] \quad \text{already look nicer so far.}$$

$$m = \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=\frac{hr}{a}}^{h} \overset{\text{density}}{K} \cdot \overset{dV}{\underline{r\,dz\,dr\,d\theta}}$$

$$= 2\pi K\int_{r=0}^{a}\left(h-\frac{hr}{a}\right)r\,dr = 2\pi K h\left(\frac{r^2}{2}-\frac{r^3}{3a}\right)\Big|_{r=0}^{a}$$

$$= 2\pi K h\left(\frac{a^2}{2}-\frac{a^3}{3a}\right) = \frac{\pi K h a^2}{3}$$

$$\Rightarrow Kh = \frac{3m}{\pi a^2} \quad ①$$

← Great! we write unknow K, h interms of m, a

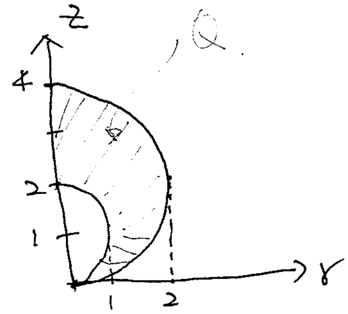
$$I_z = \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=\frac{hr}{a}}^{h} K \cdot \overset{x^2+y^2}{\underline{r^2}} \cdot r\,dz\,dr\,d\theta$$

$$= 2\pi K\int_{r=0}^{a} r^3\left(h-\frac{hr}{a}\right)dr = 2\pi K h\left(\frac{r^4}{4}-\frac{r^5}{5a}\right)\Big|_{r=0}^{a}$$

$$= 2\pi K h \cdot \frac{a^4}{20} = \frac{a^4\pi}{10}\times Kh \overset{①}{=} \frac{a^4\pi}{10}\times\frac{3m}{\pi a^2} = \boxed{\frac{3ma^2}{10}}$$

☑

#2.



big sphere : $(x)^2 + (y^2) + (z-2)^2 = 2^2$

$\Rightarrow x^2+y^2+z^2 = 2^2 z$

$\Rightarrow \rho^2 = 4\rho\cos\phi \Rightarrow \rho = 4\cos\phi$

small sphere : $x^2+y^2+(z-1)^2 = 1$

$\Rightarrow \rho^2 = 2\rho\cos\phi \Rightarrow \rho = 2\cos\phi$

So $2\cos\phi \le \rho \le 4\cos\phi$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

function = distance from the origin to (x,y,z) in Q
since we have intersection and tangent at here.

$= \rho$ ← distance from the origine.

Averge fun. $\bar{f} = \frac{1}{V}\iiint_Q f\,dV$

So we need ① Vol(Q) ② $\iiint_Q \rho\,dV$.

① $Vol(Q) = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{2\cos\phi}^{4\cos\phi} 1\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{28\pi}{3}$

② $\int_0^{2\pi}\int_0^{\pi}\int_{2\cos\phi}^{4\cos\phi} \rho\cdot\rho\sin\phi\,d\rho\,d\phi\,d\theta = 24\pi$

$\Rightarrow$ Averge distance $= \dfrac{24\pi}{\frac{28\pi}{3}} = \boxed{\dfrac{18}{7}}$