Part I

#1 $\boxed{\text{way I}}$

$4x^2+9y^2=36 \Rightarrow (2x)^2+(3y)^2=36 \leftarrow R$

Let $u=2x$, $v=3y$ $\Rightarrow$ $S: u^2+v^2=36$ $\Rightarrow$ $u=r\cos\theta,\ 0\le\theta\le 2\pi$; $v=r\sin\theta,\ 0\le r\le 6$

$\Rightarrow x=\frac{u}{2}\quad y=\frac{v}{3}\qquad J=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\begin{vmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{vmatrix}=\frac{1}{6}$

$$\iint_R 1\,dA=\iint_S \frac{1}{6}\,du\,dv=\int_{\theta=0}^{2\pi}\int_{r=0}^{6}\frac{1}{6}\,r\,dr\,d\theta=\boxed{6\pi}$$

$\boxed{\text{way II}}$:

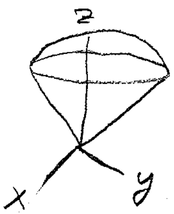
$4x^2+9y^2=36 \Rightarrow \frac{x^2}{9}+\frac{y^2}{4}=1 \Rightarrow \left(\frac{x}{3}\right)^2+\left(\frac{y}{2}\right)^2=1$

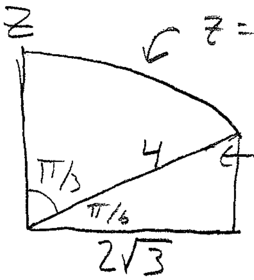
so $x=3r\cos\theta$, $y=2r\sin\theta$ $\qquad 0\le r\le 1,\ 0\le\theta\le 2\pi$

$$J=\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=\begin{vmatrix}3\cos\theta & -3r\sin\theta\\ 2\sin\theta & 2r\cos\theta\end{vmatrix}=6r\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}=6r$$

$$\text{so } \iint_R 1\,dA=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}1\cdot 6r\,dr\,d\theta=\boxed{6\pi}$$

II 1. $\iiint_Q xyz\,dV$



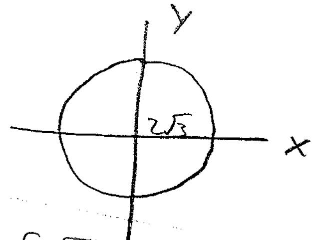


$z = \sqrt{\rho^2 - r^2} = \sqrt{16 - x^2 - y^2}$

$z = \frac{r}{\sqrt{3}} = \sqrt{\frac{x^2+y^2}{3}}$

$\boxed{\sqrt{\frac{x^2+y^2}{3}} \le z \le \sqrt{16-x^2-y^2}}$

$\hookrightarrow 0 \le r \le 2\sqrt{3}$



$\rightarrow$ $\boxed{\begin{array}{l} -2\sqrt{3} \le x \le 2\sqrt{3} \\ -\sqrt{12-x^2} \le y \le \sqrt{12-x^2} \end{array}}$

$r^2 = x^2 + y^2$
$12 = x^2 + y^2$
$y = \sqrt{12 - x^2}$

$$\boxed{\int_{x=-2\sqrt{3}}^{2\sqrt{3}} \int_{y=-\sqrt{12-x^2}}^{\sqrt{12-x^2}} \int_{z=\sqrt{\frac{x^2+y^2}{3}}}^{\sqrt{16-x^2-y^2}} xyz\,dz\,dy\,dx}$$

2. From above: $0 \le r \le 2\sqrt{3}$
$0 \le \theta \le 2\pi$

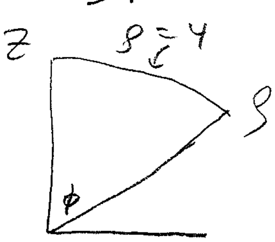
$\sqrt{\frac{x^2+y^2}{3}} \le z \le \sqrt{16-x^2-y^2}$
$r^2 = x^2+y^2$
$\sqrt{\frac{r^2}{3}} \le z \le \sqrt{16-r^2}$

$\iiint_Q xyz\,dV$

$x = r\cos\theta,\ y = r\sin\theta,\ z = z \quad dV = r\,dr\,d\theta\,dz$

$$\boxed{\int_{\theta=0}^{2\pi} \int_{r=0}^{2\sqrt{3}} \int_{z=r/\sqrt{3}}^{\sqrt{16-r^2}} r^3 \sin\theta\cos\theta\, z\,dz\,dr\,d\theta}$$

3.



$0 \le \rho \le 4$
$0 \le \phi \le \pi/3$
$0 \le \theta \le 2\pi$

$\iiint_Q xyz\,dV$

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$
conv. factor: $\rho^2\sin\phi$

$$\boxed{\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/3} \int_{\rho=0}^{4} \rho^5 \sin^3\phi\cos\phi\sin\theta\cos\theta\,d\rho\,d\phi\,d\theta}$$

Part II

#4.

$y = 2x^2+1 \Rightarrow y-2x^2=1$

$y = 2x^2+3 \Rightarrow y-2x^2=3$

$y = 2-x^2 \Rightarrow y+x^2=2$

$y = 4-x^2 \Rightarrow y+x^2=4$

Let $u = y-2x^2$, $v = y+x^2$ $\quad 1 \le u \le 3 \quad 2 \le v \le 4$

$\begin{cases} y-2x^2=u \\ y+x^2=v \end{cases} \Rightarrow \begin{matrix} 3x^2 = v-u \\ 3y = u+2v \end{matrix} \Rightarrow \begin{matrix} x = \pm\sqrt{\frac{v-u}{3}} \quad x \ge 0 \Rightarrow x = \sqrt{\frac{v-u}{3}} \\ y = \frac{u+2v}{3} \end{matrix}$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{\sqrt{3}}\cdot\frac{-1}{\sqrt{v-u}} & \frac{1}{\sqrt{3}\sqrt{v-u}} \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} = \frac{1}{\sqrt{3}\sqrt{v-u}}\left(-\frac{2}{3}-\frac{1}{3}\right) = \frac{-1}{\sqrt{3}\sqrt{v-u}}$$

$$\iint_R f(x,y)\,dA = \int_{v=2}^{4}\int_{u=1}^{3} f\left(\sqrt{\frac{v-u}{3}}, \frac{u+2v}{3}\right)\cdot\frac{-1}{\sqrt{3(v-u)}}\,du\,dv$$

Part III

First, find the plane P : $2x+y+z=2$ (first find the normal vector)
(Hint: use cross product to find the normal vector)

1.) $0 \le x \le 1$
$0 \le y \le 2-2x$
$0 \le z \le 2-y-2x$

$$V = \iiint_Q 1\, dV = \int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{2-y-2x} 1\, dz\, dy\, dx = \boxed{\frac{2}{3}}$$

2.) $\bar{x} = \frac{1}{m} \iiint_Q xk\, dV \qquad m = Vk$

$$\bar{x} = \frac{k}{Vk} \iiint_Q x\, dV = \frac{1}{2/3} \iiint_Q x\, dV = \boxed{\frac{3}{2} \int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{2-y-2x} x\, dz\, dy\, dx}$$

$$\bar{y} = \boxed{\frac{3}{2} \int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{2-y-2x} y\, dz\, dy\, dx}$$

$$\bar{z} = \boxed{\frac{3}{2} \int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{2-y-2x} z\, dz\, dy\, dx}$$

3.) $$I_x = \iiint_Q k(y^2+z^2)\, dA = k \int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{2-y-2x} (y^2+z^2)\, dz\, dy\, dx$$

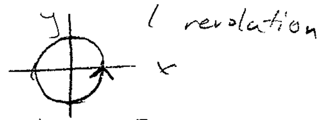
Part IV

(1.) Find center of mass of helical spring $x = \cos 2t$, $y = \sin 2t$, $z = t$, $0 \le t \le \pi$ with mass density $\sqrt{5}$.

$$x = \cos 2t \qquad dx = -2\sin 2t$$
$$y = \sin 2t \qquad dy = 2\cos 2t$$
$$z = t \qquad dz = 1$$

$$m = \int (\rho)(ds) = \int_0^{\pi} (\sqrt{5})\left(\sqrt{4\sin^2 2t + 4\cos^2 2t + 1}\, dt\right)$$
$$= \int_0^{\pi} \sqrt{5} \cdot \sqrt{5}\, dt = 5\int_0^{\pi} dt = 5(t)_0^{\pi} = 5\pi$$

By symmetry
$\Rightarrow \bar{x}, \bar{y} = 0$



$$\bar{z} = \frac{1}{m}\int (z)(\rho)(ds) = \frac{1}{5\pi}\int_0^{\pi} (t)(\sqrt{5})(\sqrt{5}\, dt) = \frac{\cancel{5}}{\cancel{5}\pi}\int_0^{\pi} t\, dt = \frac{1}{\pi}\left(\frac{1}{2}t^2\right]_0^{\pi}$$
$$= \frac{1}{2\cancel{\pi}}\left(\pi^{\cancel{2}}\right) = \frac{\pi}{2}$$

$$\boxed{\text{center of mass: } \left(0, 0, \frac{\pi}{2}\right)}$$

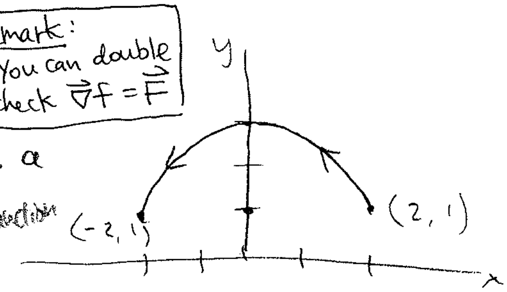
(2.) Evaluate $\vec{F} \cdot d\vec{r}$ when $\vec{F} = \langle ye^{xy}, xe^{xy} \rangle$ and C is arc on $x^2 + (y-1)^2 = 4$ from angle 0 to π

$$N_x = e^{xy} + xye^{xy}$$
$$M_y = e^{xy} + xye^{xy}$$

$\boxed{M_y = N_x}$ so we have a potential function, path independent

Remark: You can double check $\vec{\nabla} f = \vec{F}$

$\boxed{f = e^{xy}}$ is a potential function



$$\int_C \vec{F} \cdot d\vec{r} = f(b) - f(a) = f(-2,1) - f(2,1) = e^{-2} - e^2 = \boxed{\frac{1}{e^2} - e^2}$$

By Fundamental Theorem of Line integral