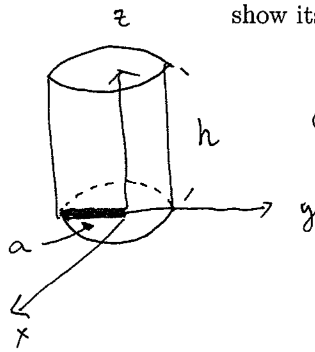
PartII (the moment of inertia around an axis)

A homogeneous cylinder has mass density 1 and radius a. If the mass is m, show its moment of inertia around its axis of symmetry equals $\frac{1}{2}a^2m$.



$\rho = 1$

① mass $= \iiint_Q \rho \, dV \underset{\because \rho = 1}{=} \iiint_Q dV$

the axis of symmetry = z-axis

⇒ the moment of inertia around its axis of symmetry

$$= I_z = \iiint_Q (x^2+y^2)\rho \, dV \underset{\because \rho=1}{=} \iiint_Q x^2+y^2 \, dV \quad ②$$

So ①

$$m = \iiint_Q dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{h} r \, dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{h} r \, dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \big[r\,z\big]_{z=0}^{h} dr\,d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} r\,h\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[\frac{r^2h}{2}\right]_{r=0}^{a} d\theta = \int_{\theta=0}^{2\pi}\frac{a^2h}{2}\,d\theta = \frac{a^2h}{2}(2\pi-0) = a^2\pi h$$

$$\Rightarrow h = \frac{m}{a^2\pi}$$

$$② \; I_z = \iiint_Q x^2+y^2 \, dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{a}\int_{z=0}^{h} r^2\cdot r\,dz\,dr\,d\theta = \frac{a^4\pi}{2}h$$

$$= \frac{a^4\pi}{2}\left(\frac{m}{a^2\pi}\right) = \frac{a^2m}{2}$$