Part 1

$\boxed{1.}$ $4x^2 + 9y^2 = 36 \implies \frac{x^2}{9} + \frac{y^2}{4} = 1$

$x = 3u$

$y = 2v$

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6$$

$S: u^2 + v^2 = 1$

Area

$$\iint_{R} 1\, dA \implies \iint_{\cancel{R}\, S} 6\, du\, dv \implies \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 6r\, dr\, d\theta = 2\pi\left[3r^2\right]_{r=0}^{1} = \boxed{6\pi}$$

different region

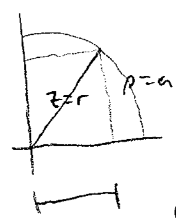
$\boxed{2.}$ Q below $\rho = a$ & above $z = r$

$\rho = a$

$\rho^2 = a^2$

$x^2 + y^2 + z^2 = a^2$

$f(x,y) = z = \sqrt{a^2 - x^2 - y^2}$



$\frac{a}{\sqrt{2}} = \frac{\sqrt{2}a}{2}$

Surface Area

$$= \iint_R \sqrt{1 + f_x^2 + f_y^2}$$

$f_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}$ $\quad (f_x)^2 = \frac{x^2}{\sqrt{a^2 - x^2 - y^2}}$

$f_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$ $\quad (f_y)^2 = \frac{y^2}{\sqrt{a^2 - x^2 - y^2}}$

$$= \iint_R \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}}$$

Plug in

$$= \iint_R \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dA = a\int_{\theta=0}^{2\pi}\int_{r=0}^{\frac{a\sqrt{2}}{2}} \frac{1}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta$$

$a^2 - r^2 = u$

$-2r\, dr = du$

$r\, dr = \frac{du}{-2}$

$r = 0$

$a^2 - 0^2 = a^2$

$r = \frac{a\sqrt{2}}{2}$

$a^2 - \left(\frac{a\sqrt{2}}{2}\right)^2 = \frac{a^2}{2}$

$$2\pi a \int_{a^2}^{\frac{a^2}{2}} u^{-\frac{1}{2}} \frac{du}{-2} = \pi a \int_{\frac{a^2}{2}}^{a^2} u^{-\frac{1}{2}}\, du = \pi a\left[2u^{\frac{1}{2}}\right]_{u=\frac{a^2}{2}}^{a^2} = \pi a\left[2a - \frac{2a}{\sqrt{2}}\right] = \boxed{2\pi a^2\left(1 - \frac{1}{\sqrt{2}}\right)}$$

Part I

$\boxed{\text{way I}}$

#1 $4x^2+9y^2=36 \Rightarrow (2x)^2+(3y)^2=36 \leftarrow R$

Let $u=2x$, $v=3y$ $\Rightarrow$ $S:\ u^2+v^2=36$ $\Rightarrow$ $u=r\cos\theta,\ 0\le\theta\le 2\pi$; $v=r\sin\theta,\ 0\le r\le 6$

$\Rightarrow x=\frac{u}{2}\quad y=\frac{v}{3}\qquad J=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\begin{vmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{vmatrix}=\frac{1}{6}$

$$\iint_R 1\,dA=\iint_S \frac{1}{6}\,du\,dv=\int_{\theta=0}^{2\pi}\int_{r=0}^{6}\frac{1}{6}\,r\,dr\,d\theta=\boxed{6\pi}$$

$\boxed{\text{way II}}$:

$4x^2+9y^2=36 \Rightarrow \frac{x^2}{9}+\frac{y^2}{4}=1 \Rightarrow \left(\frac{x}{3}\right)^2+\left(\frac{y}{2}\right)^2=1$

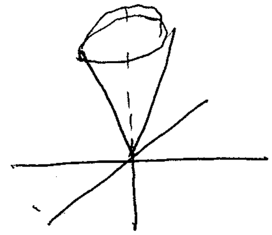
so $x=3r\cos\theta$, $y=2r\sin\theta$ $\quad 0\le r\le 1,\ 0\le\theta\le 2\pi$

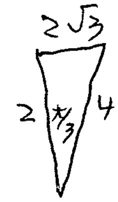
$$J=\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=\begin{vmatrix}3\cos\theta & -3r\sin\theta\\ 2\sin\theta & 2r\cos\theta\end{vmatrix}=6r\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}=6r$$

$$\text{so}\ \iint_R 1\,dA=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}1\cdot 6r\,dr\,d\theta=\boxed{6\pi}$$

Part II

$\rho = 4 \quad \phi = \frac{\pi}{3}$





1. $\iiint\limits_Q xyz\,dV$

$$-2\sqrt{3} \le x \le 2\sqrt{3}$$
$$-\sqrt{12-x^2} \le y \le \sqrt{12-x^2}$$
$$\frac{\sqrt{x^2+y^2}}{\sqrt{3}} \le z \le \sqrt{16-x^2-y^2}$$

$$\int_{x=-2\sqrt{3}}^{2\sqrt{3}} \int_{y=-\sqrt{12-x^2}}^{\sqrt{12-x^2}} \int_{z=\sqrt{\frac{x^2+y^2}{3}}}^{\sqrt{16-x^2-y^2}} xyz\,dz\,dy\,dx$$

2. $0 \le r \le 2\sqrt{3}$ $\quad x^2+y^2=r^2$ $\quad x = r\cos\theta$ $\quad dV = r\,dr\,d\theta\,dz$

$0 \le \theta \le 2\pi$ $\quad y = r\sin\theta$

$\frac{r}{\sqrt{3}} \le z \le \sqrt{16-r^2}$

$$\int_{r=0}^{2\sqrt{3}} \int_{\theta=0}^{2\pi} \int_{z=\frac{r}{\sqrt{3}}}^{\sqrt{16-r^2}} r^3 \sin\theta\cos\theta\, z\,dz\,d\theta\,dr$$

3. $0 \le \theta \le 2\pi$ $\quad x = \rho\sin\phi\cos\theta$

$0 \le \phi \le \frac{\pi}{3}$ $\quad y = \rho\sin\phi\sin\theta$

$0 \le \rho \le 4$ $\quad z = \rho\cos\phi$

$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/3} \int_{\rho=0}^{4} \rho^5 \sin^3\phi\cos\phi\,\sin\theta\cos\theta\,d\rho\,d\phi\,d\theta$$

Part II

#4.

$y = 2x^2+1 \Rightarrow y-2x^2=1$

$y = 2x^2+3 \Rightarrow y-2x^2=3$

$y = 2-x^2 \Rightarrow y+x^2=2$

$y = 4-x^2 \Rightarrow y+x^2=4$

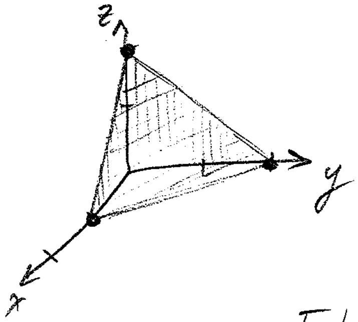
Let $u = y-2x^2$, $v = y+x^2$ $\quad 1 \le u \le 3 \quad 2 \le v \le 4$

$\begin{cases} y-2x^2=u \\ y+x^2=v \end{cases} \Rightarrow \begin{matrix} 3x^2 = v-u \\ 3y = u+2v \end{matrix} \Rightarrow \begin{matrix} x = \pm\sqrt{\frac{v-u}{3}} \quad x \ge 0 \Rightarrow x = \sqrt{\frac{v-u}{3}} \\ y = \frac{u+2v}{3} \end{matrix}$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{\sqrt{3}}\cdot\frac{-1}{\sqrt{v-u}} & \frac{1}{\sqrt{3}\sqrt{v-u}} \\ \frac{1}{3} & \frac{2}{3} \end{vmatrix} = \frac{1}{\sqrt{3}\sqrt{v-u}}\left(-\frac{2}{3}-\frac{1}{3}\right) = \frac{-1}{\sqrt{3}\sqrt{v-u}}$$

$$\iint_R f(x,y)\,dA = \int_{v=2}^{4}\int_{u=1}^{3} f\left(\sqrt{\frac{v-u}{3}}, \frac{u+2v}{3}\right)\cdot\frac{-1}{\sqrt{3(v-u)}}\,du\,dv$$

Part III Copy



formula of plane:

$P(0,1,1)$ $Q(0,2,0)$ $R(1,0,0)$

$\vec{PQ} = \langle 0,1,-1 \rangle$

$\vec{QR} = \langle 1,-2,0 \rangle$

$$\vec{PQ} \times \vec{QR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{vmatrix} = -2\hat{i} - 1\hat{j} - 1\hat{k}$$

Eqn of plane: $0 = -2(x-0) + -1(y-1) + -1(z-1)$

$$2x + y + z = 2$$

$$V = \int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{2-2x-y} 1 \, \cancel{dx\,dy\,dz} \underset{dz\,dy\,dx}{} = \frac{2}{3}$$

2. $$m = \int_0^1 \int_0^{2-2x} \int_0^{2-2x-y} K \, d_z \, d_y \, d_x = K \cdot \mathrm{Vol}(Q) = \frac{2}{3}K$$

$$\bar{x} = \frac{1}{\frac{2}{3}K} K \iiint x \, dV$$

$$= \frac{3}{2} \iiint_Q x \, dV = \frac{3}{2} \int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{2-2x-y} x \, dz\,dy\,dx$$

$$\bar{y} = \frac{3}{2} \iiint_Q y \, dV = \frac{3}{2} \int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{2-2x-y} y \, dz\,dy\,dx$$

$$\bar{z} = \frac{3}{2} \iiint_Q z \, dV = \frac{3}{2} \int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{2-2x-y} z \, dz\,dy\,dx$$

3. $$m_x = \iiint_Q k(y^2+z^2) \, dV = k \int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{2-2x-y} (y^2+z^2) \, dz\,dy\,dx$$

⋮

√

Part IV

1) $m = \int_{t=0}^{\pi} \sqrt{5}\,\sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2 + 1}\,dt$

$= \int_{t=0}^{\pi} \sqrt{5}\,\sqrt{4\sin^2 2t + 4\cos^2 2t + 1}\,dt$

$= \int_{t=0}^{\pi} \sqrt{5}\sqrt{5} = 5\pi$

$\bar{z} = \frac{1}{5\pi}\int_{t=0}^{\pi} 5z\,dt = \frac{1}{5\pi}\int_0^{\pi} 5t\,dt = \frac{1}{5\pi}\left(\frac{5t^2}{2}\right)\Big|_0^{\pi} = \left(\frac{5\pi^2}{2}\right)\left(\frac{1}{5\pi}\right) = \frac{\pi}{2}$

By symmetry, $\bar{x} = \bar{y} = 0$, so centroid $= \boxed{(0, 0, \frac{\pi}{2})}$

2) $M_y = N_x$

$e^{xy} + y e^{xy} x = e^{xy} + x e^{xy} y$ conservative

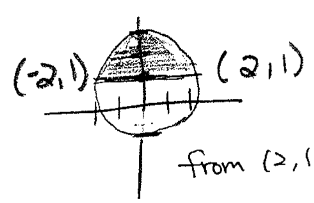
$\int y e^{xy}\,dx = e^{xy} + g(y)$

$f_y = x e^{xy} + g'(y) = x e^{xy}$

$g'(y) = 0$

$g(y) = c$

$f(x,y) = e^{xy} + c$



from (2,1) to (-2,1)

$\iint_R \langle y e^{xy}, x e^{xy} \rangle \cdot \langle dx, y\,dy \rangle\, dR$

$= e^{(-2)(1)} + c - (e^{(2)(1)} + c) = \boxed{e^{-2} - e^{2}}$