Asymptotics of pseudo-Anosov dilatations

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Outline

Background
- Definitions
- Known Results

Theorem
- Statement of Theorem
- Proof of Theorem

Definition

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Mapping class group: $Mod(S_{g,n}) = Homeo(S_{g,n})/Homeo_0(S_{g,n})$

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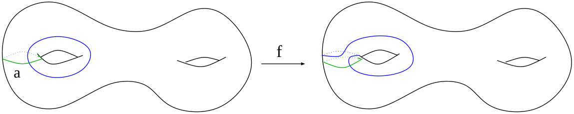
Example

$f : S_{2,0} \rightarrow S_{2,0}$ is Dehn twist along a. It's easy to check $f(a) = a$, so f is reducible.



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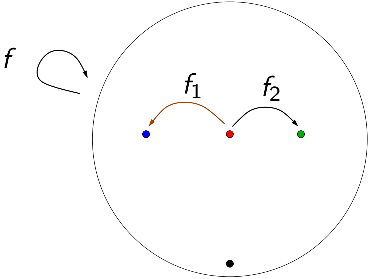

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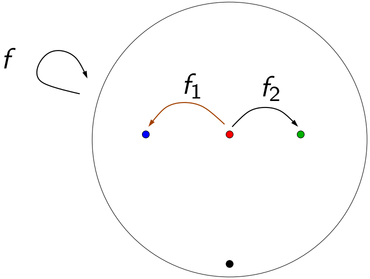

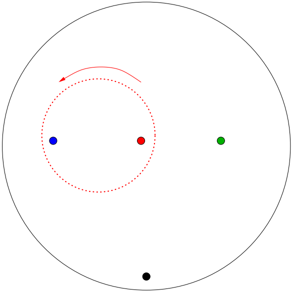

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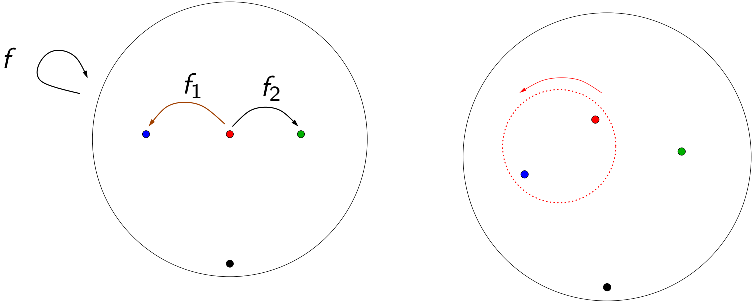

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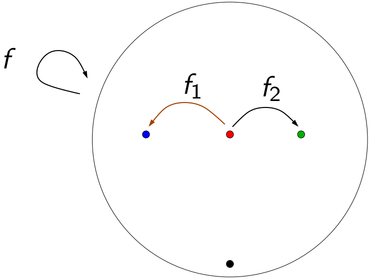

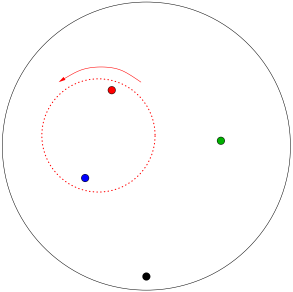

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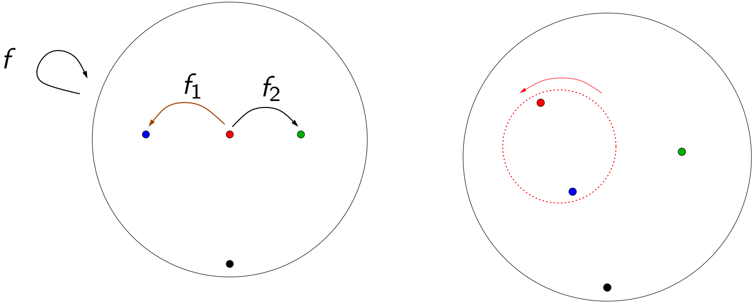

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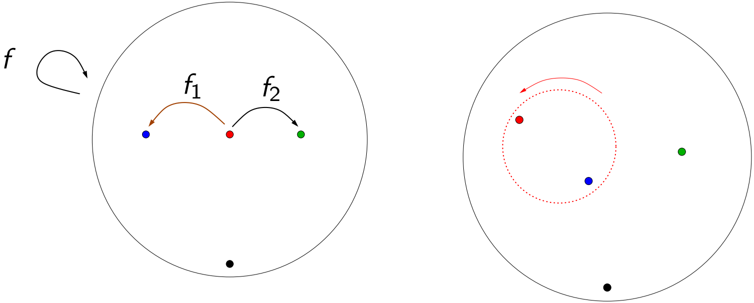

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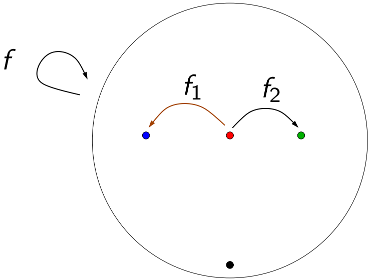

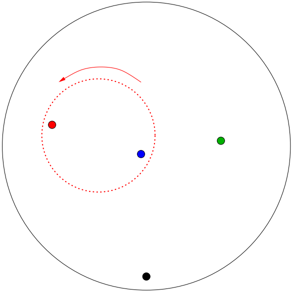

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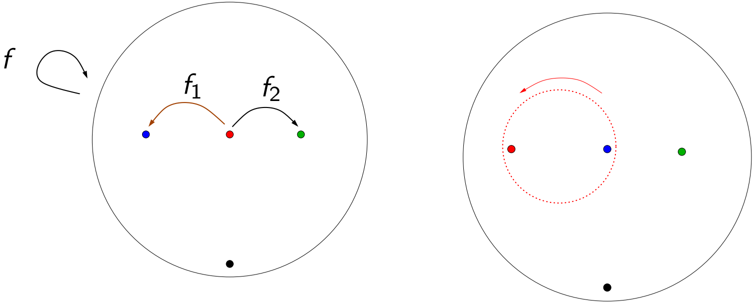

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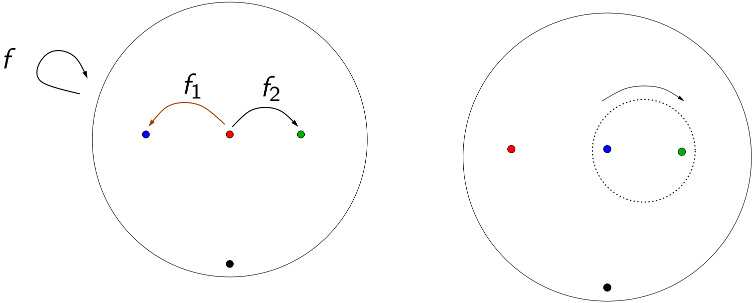

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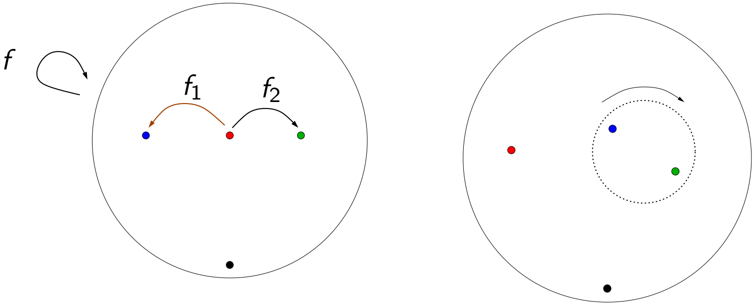

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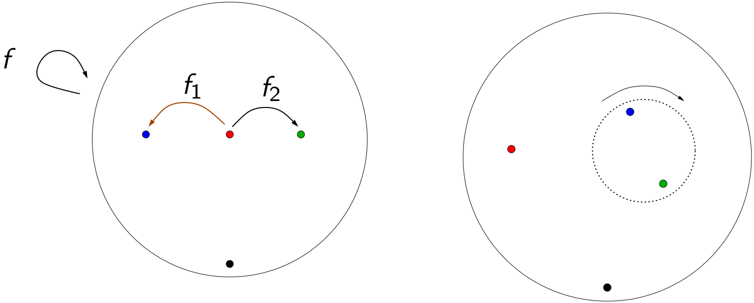

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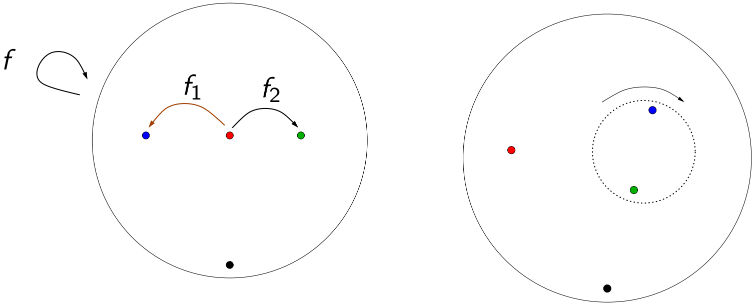

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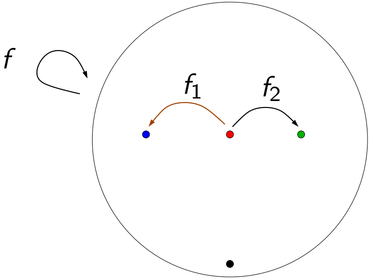

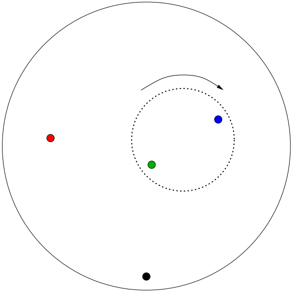

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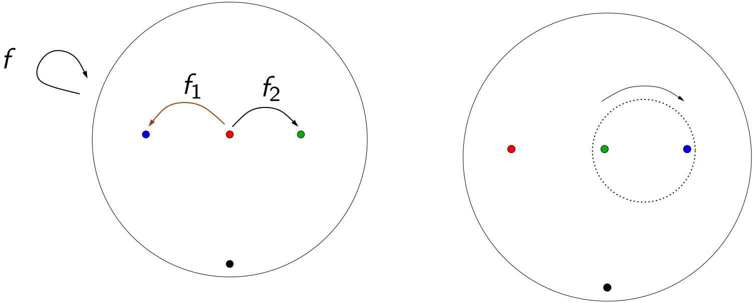

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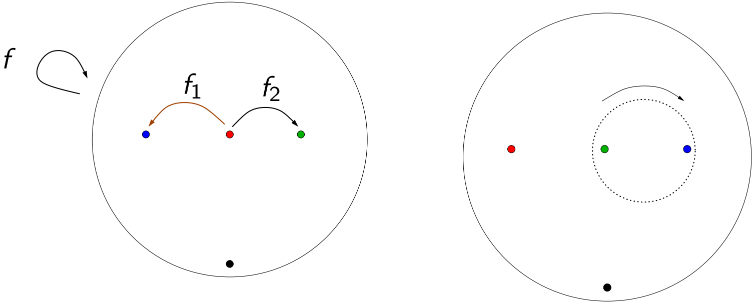

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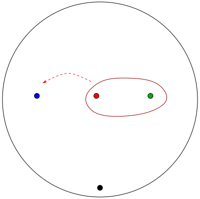

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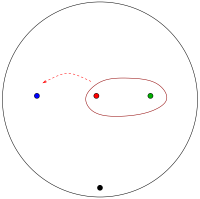

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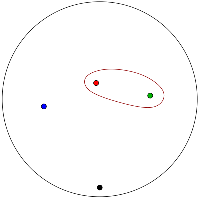

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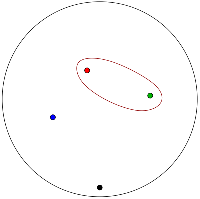

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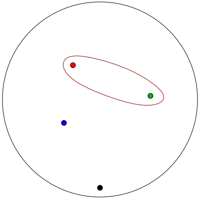

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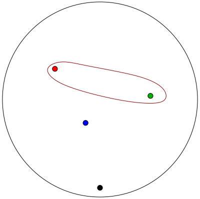

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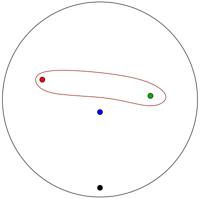

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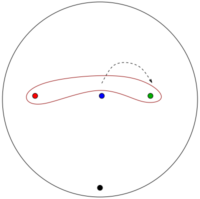

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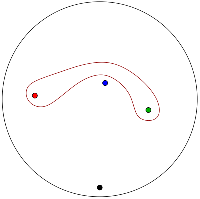

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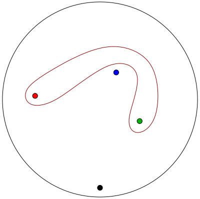

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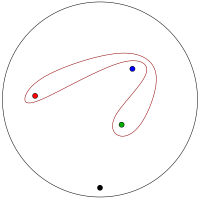

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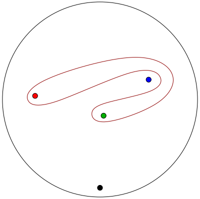

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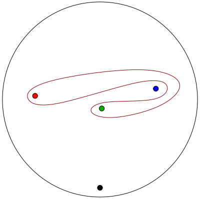

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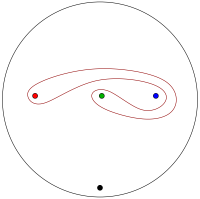

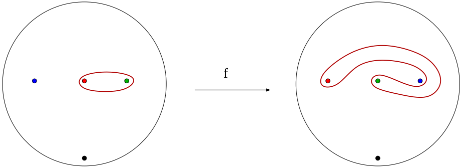

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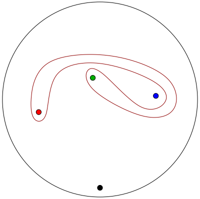

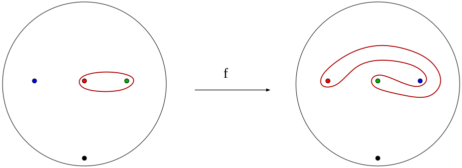

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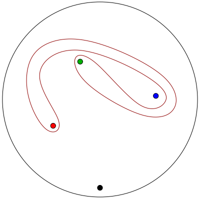

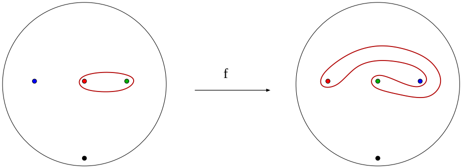

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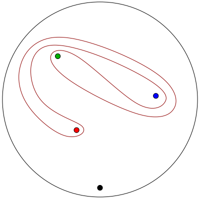

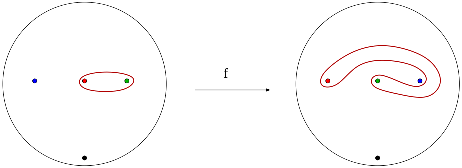

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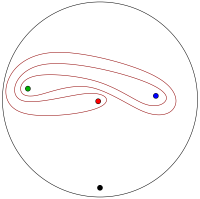

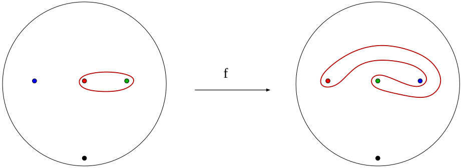

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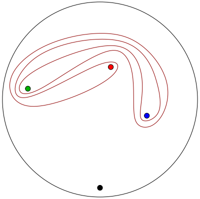

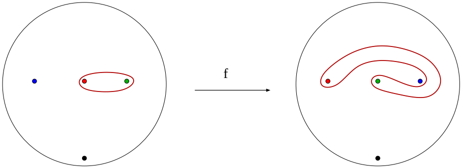

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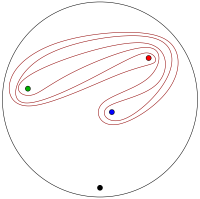

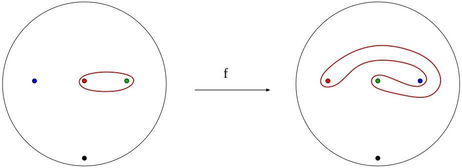

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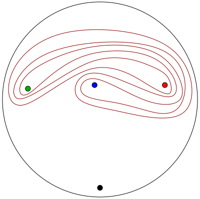

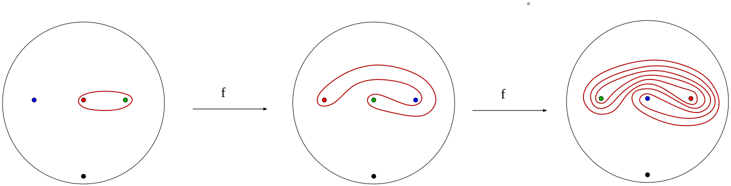

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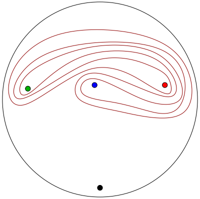

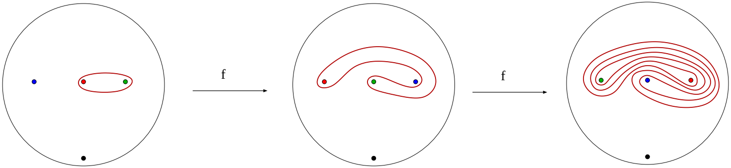

pseudo-Anosov dilatation

Theorem (Thurston)

Given any hyperbolic metric on S, $f \in Mod(S_{g,n})$ *pseudo-Anosov, there exists* $\lambda(f) > 1$,

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Definition

$\lambda(f) :=$ the dilatation of pseudo-Anosov $f \in Mod(S_{g,n})$.

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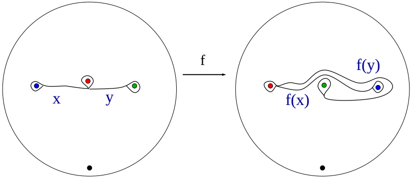

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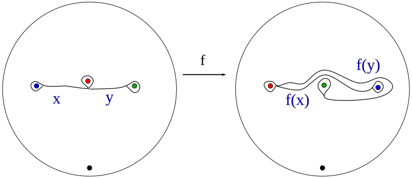

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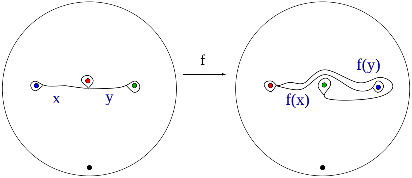

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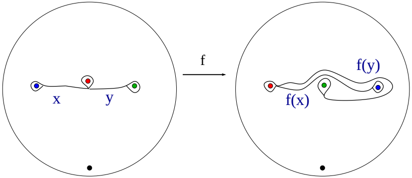

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Remark

M is a Perron-Frobenius matrix, and the Perron-Frobenius eigenvalue is the dilatation.

least pseudo-Anosov dilatation

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Question: (Penner)

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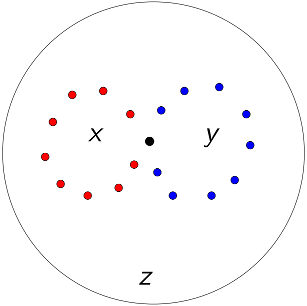

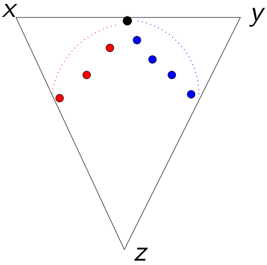

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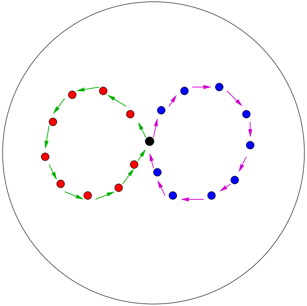

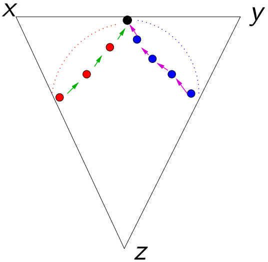

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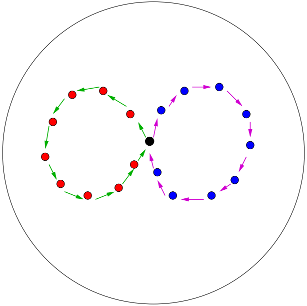

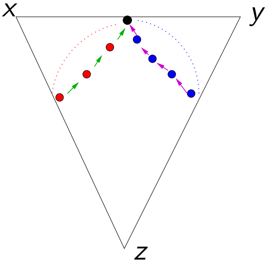

where $log\lambda(f) \leq \frac{3\log n}{n}$

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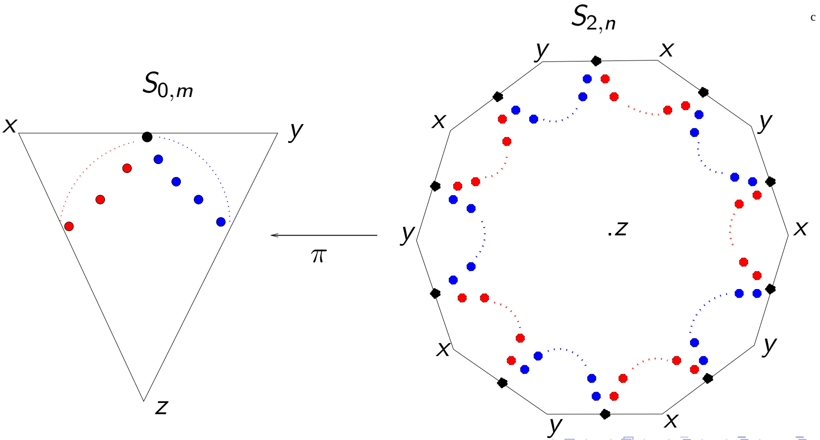

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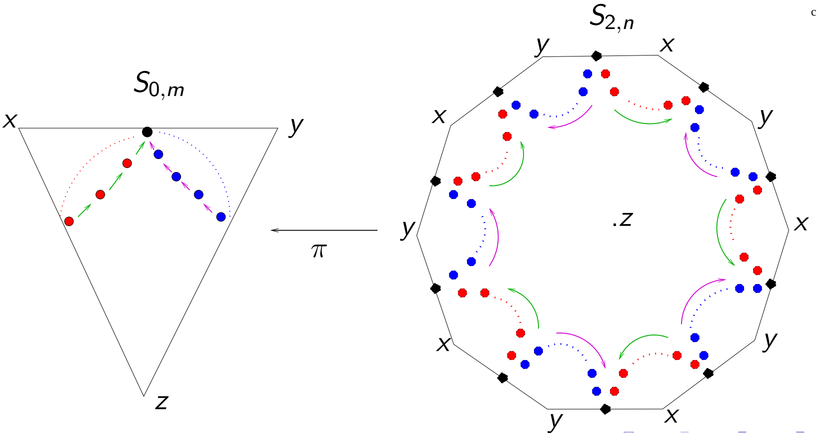

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This lower bound only depends on g.

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$\Rightarrow \log \lambda(f) \geq \frac{\log k}{2\alpha k} \geq \frac{\log(6g+3n-6)}{2\alpha(6g+3n-6)}$.

How does $l_{g,n}$ behave?

g	n	**The asymptotic behavior of $l_{g,n}$**
0	≥ 3	$1/n$
1	even	$1/n$
> 1	$0, 1, 2, 3, 4$	$1/g$
> 1	$g, g+1, g+2$	$1/g$
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Question

What are asymptotic behaviors of $l_{g,n}$ along different (g, n)-rays in the (g, n) plane?