Asymptotics of pseudo-Anosov dilatations

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Plan for Today

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Mapping class group: $Mod(S) = Homeo(S)/Homeo_0(S)$

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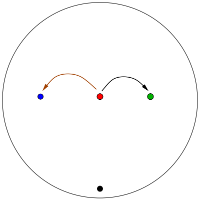
Nielsen-Thurston Classification

A mapping class is either periodic, reducible, or pseudo-Anosov.

Thurston's example:

$f : S_{0,4} \longrightarrow S_{0,4}$ (Red then Black) is pseudo-Anosov.



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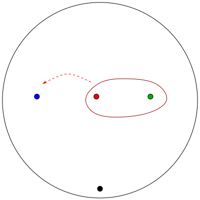

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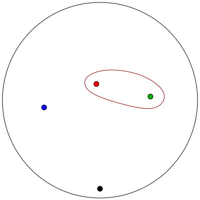

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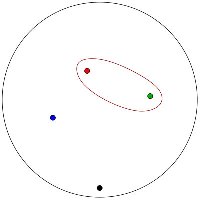

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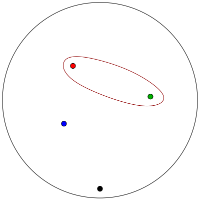

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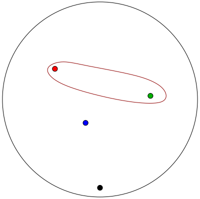

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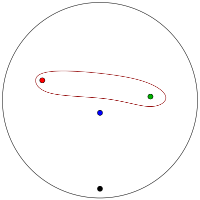

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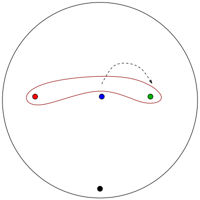

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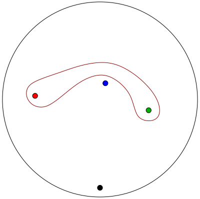

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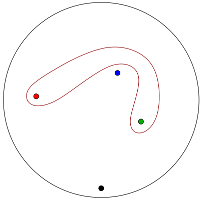

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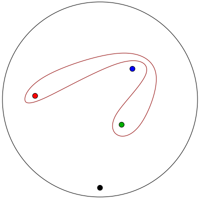

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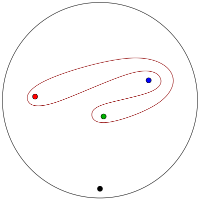

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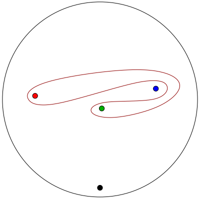

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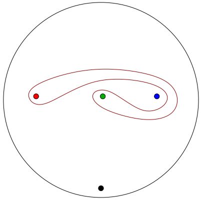

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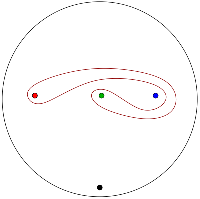

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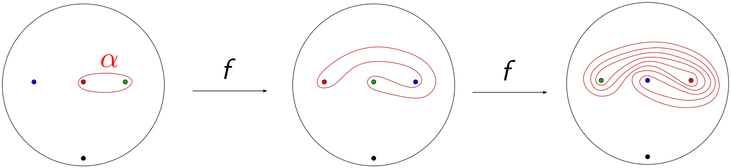

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Theorem (Thurston)

Given any hyperbolic metric on S, $f \in Mod(S)$ pseudo-Anosov,

$$\lim_{k \to \infty} \sqrt[k]{\ell([f^k(\alpha)])} = \lambda(f),$$

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$l_{g,n} := \min\{\log\lambda(f) | f \in Mod(S_{g,n}) \text{ pseudo-Anosov}\}$.
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(least pseudo-Anosov dilatation)

Known results on $l_{g,n} = min\{\log \lambda(f) | f \in Mod(S_{g,n}) \text{ pseudo-Anosov}\}$

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  Combining with Penner's lower bound, $\Rightarrow \frac{\log 2}{4n-12} \leq l_{0,n} < \frac{2\log(2+\sqrt{3})}{n-3}$, for $n \geq 4$.

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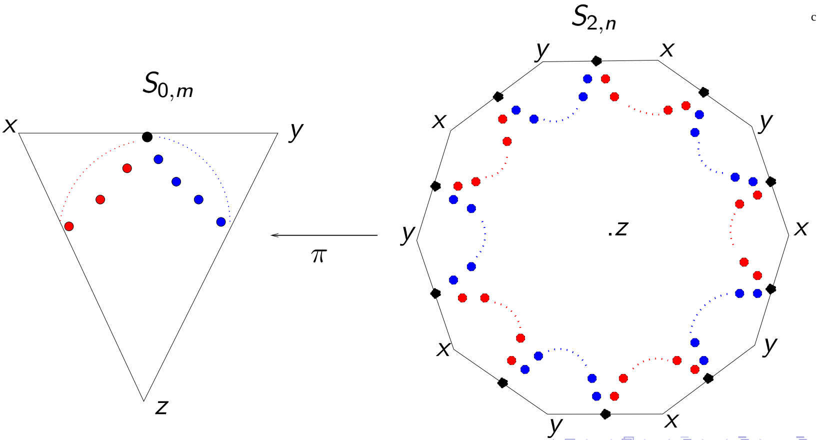

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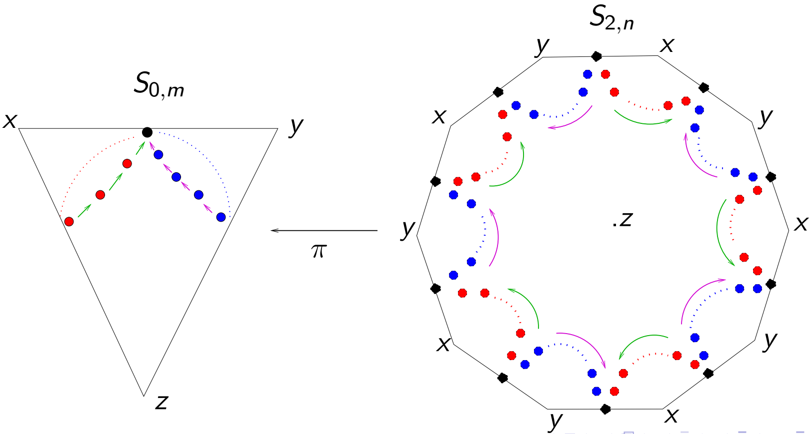

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This lower bound only depends on g.

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$\Rightarrow \log \lambda(f) \geq \frac{\log n}{\alpha n}$.