# Bounds on least pseudo-Anosov dilatations

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# Plan for Today

# Definitions

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$$f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda\mu^s),$$
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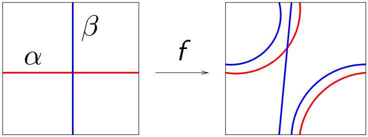
for $\lambda > 1$.
We call $\lambda = \lambda(f)$ the **dilatation** of $f$.

# Example of Pseudo-Anosov

Let $f$ be a homeomorphism of $S_{1,0}$ such that $f(\alpha) = \alpha + \beta$ and $f(\beta) = \alpha + 2\beta$.



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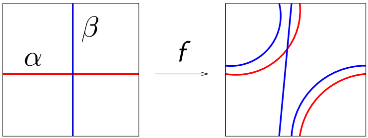


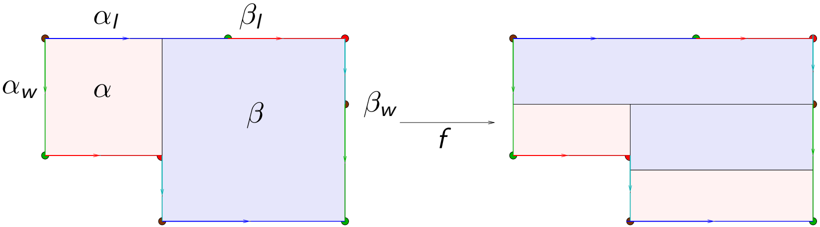

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$\leftrightarrow$ its leading eigenvalue equals $\lambda(f)$, the dilatation of $f$.

$$\lambda(f) = \frac{3+\sqrt{5}}{2}$$

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Recall $\lambda(f)$ = the dilatation of $f$.

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**Question**
What is $l_{g,n}$?

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  Combining with Penner's lower bound,
  $\Rightarrow \frac{\log 2}{4n-12} \leq l_{0,n} < \frac{2\log(7+\sqrt{45})}{n-1}$, for $n \geq 4$.

# Main Theorem

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## Theorem (T,2008)

*Given genus $g \geq 2$, $\exists$ $c_g$, a constant depending on $g$, such that*

$$\frac{\log n}{c_g n} < l_{g,n} < \frac{c_g \log n}{n}, \ \forall n \geq 3.$$

# The proof of the lower bound $\log \lambda(f) > \frac{\log n}{c_g n}$

**Proof**

Given $g \geq 2$ and $f : S_{g,n} \to S_{g,n}$ pseudo-Anosov.

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## Proof

Given $g \geq 2$ and $f : S_{g,n} \to S_{g,n}$ pseudo-Anosov.
*(Bestvina-Handel)* $\exists$ a Markov partition of $k \leq 18g + 6n - 18$ rectangles with a matrix $M$ s.t. the leading eigenvalue of $M$ equals $\lambda(f)$.

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Suppose Lefschetz number $L(f) < 0$. This tells us that: $\exists$ $R$ s.t. $int(f(R)) \cap int(R) \neq \emptyset$.

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$$\Rightarrow \log \lambda(f) \geq \frac{\log k}{2k} > \frac{\log(18g + 6n - 18)}{2(18g + 6n - 18)}$$

# the intersection of $R$

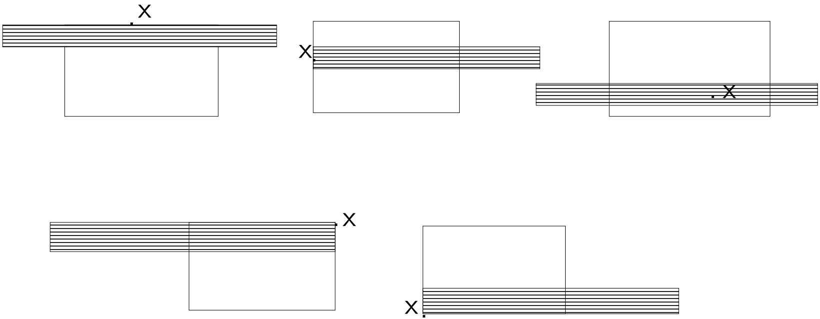


Figure: $x$ is a fixed point of $f$

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- a pseudo-Anosov on a connected subsurface
- the identity
- a multitwist

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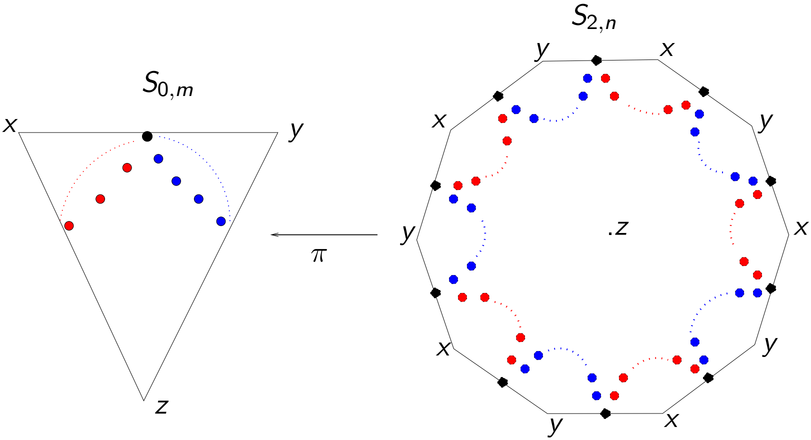
By choosing $c_g$ wisely, we will have the desired bound.

# The proof of the upper bound

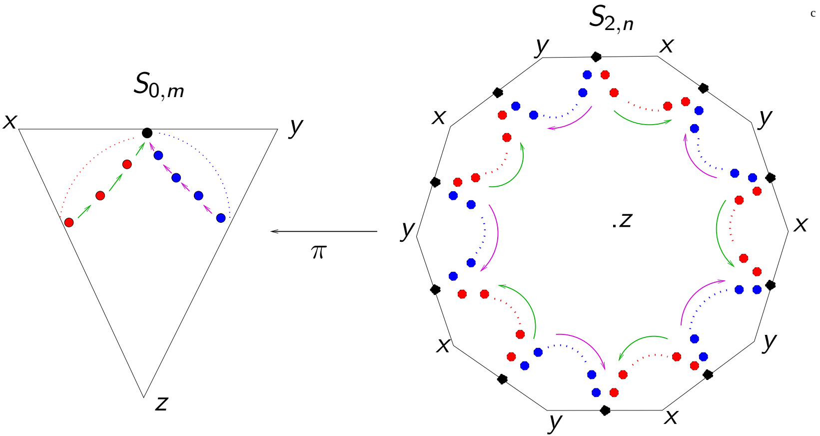
## Main Idea

Take a branched cover over H-K's example



# The proof of the upper bound

The pseudo-Anosov homeomorphism $f$ of $S_{0,m}$ and its lift pseudo-Anosov homeomorphism $\widetilde{f}$ of $S_{2,n}$.



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Therefore, $l_{2,n} < \frac{10}{n-1} \log\left(\frac{n-1}{5}\right)$, where $n = 5m-4$.

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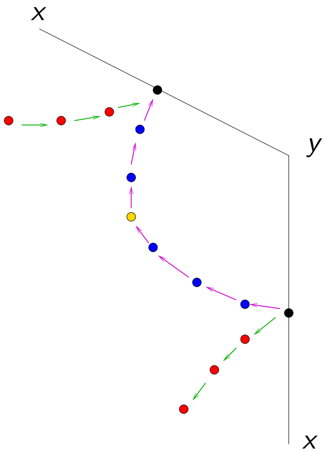
Therefore, $l_{2,n} < \frac{10}{n-1} \log\left(\frac{n-1}{5}\right)$, where $n = 5m-4$.
Similarly, take a different branched cover $S_{g,n}$ of any genus $> 2$.
Then we have

$$l_{g,n} < \frac{4g+2}{n-1} \log\left(\frac{n-1}{2g+1}\right)$$

where $n = (2g+1)(m-1)+1$.

# The proof of the upper bound

Adding a marked point (yellow) into $S_{g,n}$.



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What is the right asymptotic behavior of $l_{g,n}$ for any $g$ and $n$?

# Thank you!