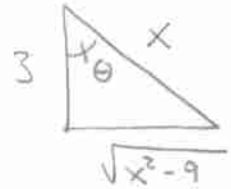
$\boxed{1.}$ Recall: $\int \sec^n x\,dx = \dfrac{\sec^{n-2}x\tan x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2}x\,dx$

and $\int \sec x\,dx = \ln|\sec x + \tan x| + C.$

(I would give this to you on the test.)

$$\int \sqrt{x^2-9}\,dx$$



$x = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$\tan\theta = \dfrac{\sqrt{x^2-9}}{3}$

$3\tan\theta = \sqrt{x^2-9}$

$$= \int 9\sec\theta\tan^2\theta\,d\theta$$

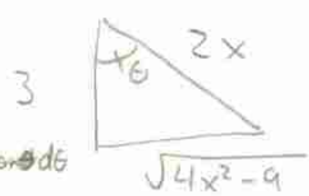
$$= 9\int \sec^3\theta - \sec\theta\,d\theta$$

$$= 9\left(\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\int\sec\theta\,d\theta - \int\sec\theta\,d\theta\right)$$ by above reduction formula

$$= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{9}{2}\left(\frac{x}{3}\cdot\frac{\sqrt{x^2-9}}{3} - \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|\right) + C.$$

$\boxed{2.}$ $\displaystyle\int \frac{x^2}{\sqrt{4x^2-9}}\,dx$



$\sec\theta = \dfrac{2x}{3}$

$\frac{3}{2}\sec\theta = x$

$dx = \frac{3}{2}\sec\theta\tan\theta\,d\theta$

$x^2 = \frac{9}{4}\sec^2\theta$

$\cot\theta = \dfrac{3}{\sqrt{4x^2-9}}$

$\frac{1}{3}\cot\theta = \dfrac{1}{\sqrt{4x^2-9}}$

$$= \int \frac{9}{4}\sec^2\theta\cdot\frac{1}{3}\cot\theta\cdot\frac{3}{2}\sec\theta\tan\theta\,d\theta$$

$$= \frac{9}{8}\int \sec^3\theta\,d\theta$$

$$= \frac{9}{8}\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + c\right)$$

$$= \frac{9}{16}\left(\frac{2x}{3}\cdot\frac{\sqrt{4x^2-9}}{3} + \frac{1}{2}\ln\left|\frac{2x}{3} + \frac{\sqrt{4x^2-9}}{3}\right| + c\right)$$

$\boxed{3.}$ $\displaystyle\int \frac{x^4+1}{x^2+1}\,dx$

$$\begin{array}{r|l} & x^2-1 \\ \hline x^2+1 & x^4+1 \\ & -(x^4+x^2) \\ \hline & -x^2+1 \\ & -(-x^2-1) \\ \hline & 2 \end{array}$$

$$= \int x^2-1+\frac{2}{x^2+1}\,dx$$

$$= \frac{1}{3}x^3 - x + 2\arctan x + C.$$

4. $\int \sec^3 x \tan^3 x \, dx$

$= \int \sec x \tan x \left(\sec^2 x\left(\sec^2 x - 1\right)\right) dx$

$= \int u^4 - u^2 \, du$ $\quad$ $u = \sec x$, $du = \sec x \tan x \, dx$

$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$

5. $\int \frac{4x^3 - x + 1}{x^3+1} dx = \int 4 - \frac{x+3}{x^3+1} dx$

$$\begin{array}{r} 4 \\ x^3+1 \overline{\big) 4x^3 - x + 1} \\ -(4x^3+4) \\ \hline -x-3 \end{array}$$

Partial Fractions: $\frac{x+3}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

$x + 3 = A(x^2 - x + 1) + (Bx + C)(x+1)$

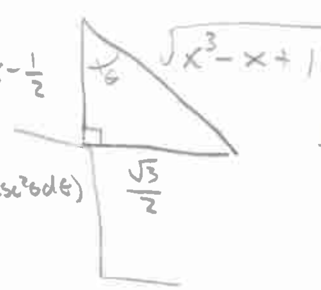
$A + B = 0$

$-A + B + C = 1$ $\quad \Rightarrow \quad$ $C = 1 + 2A$, $A = \frac{2}{3}$, $C = \frac{7}{3}$, $B = -\frac{2}{3}$

$A + C = 3$

So $\int \frac{x+3}{x^3+1} dx = \frac{1}{3}\int \frac{2}{x+1} - \frac{2x-7}{x^3-x+1} dx$

$\int \frac{2x-7}{x^3-x+1} dx = \int \frac{2x-7}{(x-\frac{1}{2})^2 + \frac{3}{4}} dx$



$\frac{\sqrt{3}}{2}\cot\theta = x - \frac{1}{2}$

$x = \frac{\sqrt{3}}{2}\cot\theta + \frac{1}{2}$

$dx = -\frac{\sqrt{3}}{2}\csc^2\theta \, d\theta$

$2x - 7 = \sqrt{3}\cot\theta - 6$

$\frac{1}{x^3-x+1} = \left(\frac{2}{\sqrt{3}}\right)^2 \sin^2\theta$

$= \int \left(\sqrt{3}\cot\theta - 6\right)\left(\frac{2}{\sqrt{3}}\right)^2 \sin^2\theta \left(-\frac{\sqrt{3}}{2}\csc^2\theta \, d\theta\right)$

$= -\frac{2}{\sqrt{3}}\int \sqrt{3}\cot\theta - 6 \, d\theta$

$= -2\ln|\sin\theta| + \frac{12}{\sqrt{3}}\theta + C$

$= -2\ln\left|\frac{\frac{\sqrt{3}}{2}}{\sqrt{x^3-x+1}}\right| + \frac{12}{\sqrt{3}}\operatorname{arccot}\left(\frac{2}{\sqrt{3}}\left(x - \frac{1}{2}\right)\right) + C$

$I = 4x + \frac{2}{3}\ln|x+1| + 2\ln\left|\frac{\frac{\sqrt{3}}{2}}{\sqrt{x^3-x+1}}\right| + \frac{12}{\sqrt{3}}\operatorname{arccot}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right) + C$