

# Galois actions on fundamental groups.

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0. Let  $V := \mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$ . Let us fix a prime  $l$ . Let  $\pi_1(V_{\mathbb{Q}}; \vec{o}_1)$  be the maximal pro- $l$  quotient of the étale fundamental group of  $V_{\mathbb{Q}}$ . The tangential base point  $\vec{o}_1$  corresponds taking fiber over

$$\text{Spec } \Omega = \text{Spec} \left( \bigcup_{n=1}^{\infty} \overline{\mathbb{Q}}((z^{1/n})) \right) \longrightarrow \text{Spec } \overline{\mathbb{Q}}[z, \frac{1}{z}, \frac{1}{z^2}, \dots]$$

Laurent power series in  $z^{1/n}$

The Galois group  $G_{\mathbb{Q}}$  acts on  $\pi_1(V_{\mathbb{Q}}; \vec{o}_1)$ . This action was studied by Ihara, Deligne, Grothendieck. I shall follow mainly approach of Ihara. In this talk I would like to say something about action of Galois group on  $\pi_1(\mathbb{P}_{\mathbb{Q}}^1 \setminus (\{0, \infty\} \cup \mu_8); \vec{o}_1)$ .

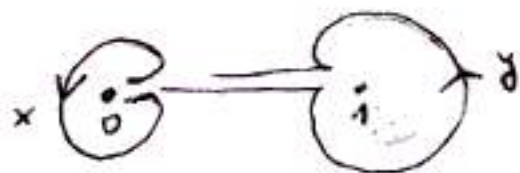
## 1. $l$ -adic Galois polylogarithms.

Let  $K$  be a number field and let  $z \in V(K)$ . Let  $\gamma$  be a path from  $\vec{o}_1$  to  $z$  on  $V_{\mathbb{Q}}$  (natural isomorphism of fiber functors over  $\vec{o}_1$  and over  $z$ ). For any  $\delta \in G_K$

we consider a loop

$$f_{\gamma}(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) = \gamma^{-1} \circ (\sigma \circ \gamma \circ \sigma^{-1}) \in$$

$\pi_1(V_{\mathbb{K}}; \vec{o}_1)$  - free pro-l group on  $x$  and  $y$



$$f_{\gamma}(\sigma) \equiv x^{\alpha(\sigma)} \cdot y^{\beta(\sigma)} \pmod{\Gamma^2 \pi_1(V_{\mathbb{K}}; \vec{o}_1)}$$

We want to know coefficients  $\alpha(\sigma)$  and  $\beta(\sigma)$ .

Observe that

$$x: z^{1/n} \rightarrow \sum_{i=1}^n z^{i/n} \cdot z^{1/n}$$

$z$ -local parameters corresponding to  $\vec{o}_1$   
 $x$ : monodromy along  $x$

$$y: z^{1/n} \rightarrow z^{1/n}$$

Hence to calculate  $\alpha(\sigma)$  we need to act

$$f_{\gamma}(\sigma): z^{1/n} \xrightarrow{\sigma^{-1}} z^{1/n} \xrightarrow{\gamma} (z + z^{-2})^{1/n} = z^{1/n} \left(1 + \frac{z^{-2}}{z}\right)^{1/n} = z^{1/n} \left(1 + \binom{1/n}{1} \frac{z^{-2}}{z} + \dots\right) \xrightarrow{\sigma} \frac{\sigma(z^{1/n})}{z^{1/n}} z^{1/n} \xrightarrow{\sigma^{-1}} \sum_{i=1}^{K(z)(\sigma)} z^{i/n} \cdot z^{1/n}$$

hence  $f_{\gamma}(\sigma) \equiv x^{K(z)(\sigma)} \cdot y^{K(1-z)(\sigma)} \pmod{\Gamma^2 \pi_1(V_{\mathbb{K}}; \vec{o}_1)}$

Let

$k: \pi_1(V_{\mathbb{K}}; \vec{o}_1) \rightarrow \mathbb{Q}_2 \langle\langle X, Y \rangle\rangle$  - formal power series in non-commutative variables  $X$  and  $Y$

be a continuous multiplicative embedding given by  $k(x) := e^X$  and  $k(y) := e^Y$ . Then

$\log(k(\prod_y^{\delta}(c)))$  is a Lie element (infinite sum of Lie elements). Let us define

$l_k(z)$  by the following congruence

$$\log k(\prod_y^{\delta}(c)) \equiv k(z)_y X + k(1-z)_y Y + \sum_{k=2}^{\infty} \frac{l_k(z)_y}{k} [Y, X^{k-1}]$$

mod  $I_2$ ,

$I_2$  is a Lie ideal generated by Lie brackets with 2 or more  $Y$ 's.

$l_k(z)_y : G_K \rightarrow \mathbb{Q}(k)$  have all functional properties of classical polylogarithms.

Proposition (Nakamura+Zw) Let us write

$$\log k(x^{-K(z)_y} \prod_y^{\delta}(c)) \equiv \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} l_k(z)_y [Y, X^{k-1}] \text{ mod } I_2.$$

Then

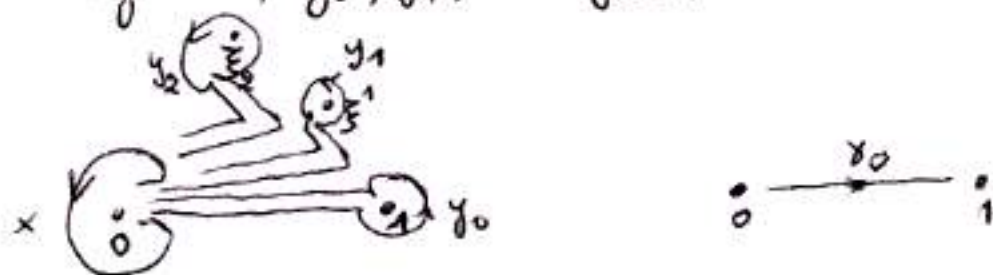
$$\sum_{k=1}^m \frac{l_k(z)_y}{k} Y^k = \frac{\prod_{i=1}^{m-1} (1 - \sum_{l=1}^{i-1} z^l)^{\frac{i-k-1}{i}}}{\left( \prod_{i=1}^{m-1} (1 - \sum_{l=1}^{i-1} z^l)^{\frac{i-k-1}{i}} + K(z)_y \frac{1}{z^m} \right)^{\frac{i-k-1}{i}}}. \quad \text{(Gabber)}$$

Observe that for  $z=1$  and  $z = \prod_m^k$  one gets (linear combinations of) Soulé classes. (These are classes in  $H^1(G_{\mathbb{Q}}, \mathbb{Z}(k)) \rightarrow H^1(G_{\mathbb{Q}(m)}, \mathbb{Z}(k))$ ).

$$2. \frac{\mathbb{P}^1_{\mathbb{Q}(\mu_n)} \setminus (\{0, \infty\} \cup \mu_n)}{\mathbb{Q}(\mu_n)}$$

Let  $V := \mathbb{P}^1_{\mathbb{Q}(\mu_n)} \setminus (\{0, \infty\} \cup \mu_n)$ . The group

$\pi_1 := \pi_1(V_{\mathbb{Q}}; \vec{o}_1)$  is free, pro- $\ell$  freely generated by  $x, y_0, y_1, \dots, y_{n-1}$ .



Let  $k: \pi_1(V_{\mathbb{Q}}; \vec{o}_1) \rightarrow \mathbb{Q}_\ell\langle\{X, Y_0, \dots, Y_{n-1}\}\rangle$  be given by  $k(x) = e^X$ ,  $k(y_i) = e^{Y_i}$ .

With the action of  $G_{\mathbb{Q}(\mu_n)}$  on  $\pi_1$  there is associated a filtration  $\{G_k\}_{k \in \mathbb{N}}$  of  $G_{\mathbb{Q}(\mu_n)}$

$$G_k := \ker(G_{\mathbb{Q}(\mu_n)} \rightarrow \text{Aut}(\pi_1 / \Gamma^{k+1} \pi_1))$$

Observe that  $G_1 = G_{\mathbb{Q}(\mu_n)}$ .

The action of  $G_{\mathbb{Q}(\mu_n)}$  on  $\pi_1$  induces a representation

$$\rho_{\vec{o}_1}: G_{\mathbb{Q}(\mu_n)} \rightarrow \text{Aut}(\mathbb{Q}_\ell\langle\{X, Y_0, \dots, Y_{n-1}\}\rangle)$$

Passing to Lie algebras and then to associated graded Lie algebras we get a

morphism of Lie algebras

$$\Phi_{01} : \bigoplus_{k=1}^{\infty} G_k / G_{k+1} \otimes \mathbb{Q} \rightarrow \text{Der}_{\mathbb{Z}/n}^* (\text{Lie}(X, Y_0, \dots, Y_{n-1}))$$

$\{ D \in \text{Der}(\text{Lie}(\dots)) \mid \exists \beta(X, Y_0, \dots, Y_{n-1}) \in \text{Lie}(\dots), \forall i,$

$$D(Y_i) = [Y_i, \beta(X, Y_0, \dots, Y_{n-1})] \text{ and } D(X) = 0$$

Such  $D$  we denote by  $D_\beta$

$$\text{Der}_{\mathbb{Z}/n}^* (\text{Lie}(\dots)) \simeq (\text{Lie}(X, Y_0, \dots, Y_{n-1}) / \langle Y_0 \rangle, \{ \})$$

$$\{ A, B \} := [A, B] + D_A(B) - D_B(A)$$

Lemma 2.1. Let  $\phi \in G_k$ . Then

$$\Phi_{01}(\phi) \equiv \sum_{i=0}^{n-1} \ell_k \left( \sum_n^{n-i} \right) (\phi) [Y_i, X^{k-1}] \pmod{I_2} \text{ for } k > 1$$

and

$$\Phi_{01}(\phi) = \sum_{i=1}^{n-1} \ell(1 - \sum_n^{n-i}) (\phi) Y_i \text{ for } k=1$$

Let  $V = \mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0, \infty\} \cup \mu_8)$ . We want

to describe image of  $\Phi_{01}$ . We shall use an idea from Deligne talk in to work modulo 2, Deligne's talk concerned  $\mathbb{P}^1 \setminus \{0, 1, -1, \infty\}$ .

Lemma 2.2. In the image of

$$\Phi_{01}^{\rightarrow} : \bigoplus_{k=1}^{\infty} G_k / G_{k+1} \otimes \mathbb{Q} \rightarrow (\text{Lie}(X, Y_0, \dots, Y_7) / \langle Y_0 \rangle, \{ \})$$

there are homogenous elements

$$D_k^1, D_k^5 \text{ of degree } k \text{ for } k=1, 2, \dots$$

such that

$$D_k^1 \equiv 2^{2(k-1)} \binom{k-1}{1+(-1)^{k-1}} Y_0 X^{k-1} + (1-2^{k-1}) Y_1 X^{k-1} + 2^{k-1} (1-2^{k-1}) Y_2 X^{k-1} \\ + 2^{k-1} \binom{k-1}{1+(-1)^{k-1}} (1-2^{k-1}) Y_4 X^{k-1} + 2^{k-1} (1-2^{k-1}) (-1)^{k-1} Y_6 X^{k-1} + \\ (-1)^{k-1} (1-2^{k-1}) Y_7 X^{k-1} \pmod{I_2}$$

$$D_k^5 \equiv \dots (1-2^{k-1}) Y_3 X^{k-1} \\ (-1)^{k-1} (1-2^{k-1}) Y_5 X^{k-1} \dots \pmod{I_2}$$

for  $k > 1$

and

$$D_1^1 = Y_1 + Y_2 + 2Y_4 + Y_6 + Y_7,$$

$$D_1^5 = Y_2 + Y_3 + 2Y_4 + Y_5 + Y_6.$$

Proof. One uses distribution and inversion relations for polylogarithms, the fact that  $\dim H^1(G_{\mathbb{Q}(u_8)}; \mathbb{D}_1(k)) = 2$  for  $k > 1$  and that  $\lambda_k(\zeta_2^1)$  and  $\lambda_k(\zeta_8^3)$  generate this group.

In degree 1 one uses the fact that

$$1 - \zeta_8^2 = (1 - \zeta_8^1)(1 - \zeta_8^3) \cdot \zeta_8^{-3}$$

and

$$1 - \zeta_8^4 = (1 - \zeta_8^1)^2 (1 - \zeta_8^3)^2 \cdot \zeta_8^2.$$

Theorem 2.3. The image of  $\Phi_{01}^{\mathbb{Z}_8}$  is a free Lie algebra freely generated by elements  $D_1^1, D_1^5, D_2^1, D_2^5, \dots, D_k^1, D_k^5, \dots$ .

Proof. Call elements on the right hand side of congruences of Lemma 2.2  $y_k^1$  and  $y_k^5$  ( $k \in \mathbb{N}$ ).

Then

$$\{ \dots D_{i_1}^{E_1}, \dots, D_{i_r}^{E_r} \} \equiv \{ y_{i_1}^{E_1}, \dots, y_{i_r}^{E_r} \} \pmod{I_{r+1}}.$$

Let us set  $S_k^1 = Y_1 X^{k-1}$  and  $S_k^5 = Y_5 X^{k-1}$  for  $k \in \mathbb{N}$ .

Let  $J = J(Y_0, Y_2, Y_3, Y_4, Y_6, Y_7)$  be a Lie ideal of  $\text{Lie}(X, Y_0, \dots, Y_7; \mathbb{Z}/2)$  generated by  $Y_0, Y_2, Y_3, Y_4, Y_6, Y_7$ .

Then

$$\text{Lie}(X, Y_0, \dots, Y_7; \mathbb{Z}/2) = \text{Lie}(X, Y_1, Y_5) \oplus J,$$

One shows that

$$\{ \dots y_{i_1}^{E_1}, \dots, y_{i_r}^{E_r} \} \equiv [ \dots S_{i_1}^{E_1}, \dots, S_{i_r}^{E_r} ] \pmod{J}.$$

The elements  $S_k^1, S_k^5, k \in \mathbb{N}$  generate freely a free Lie subalgebra of  $\text{Lie}(X, Y_1, Y_5)$ , hence basic Lie elements in  $S_k^1, S_k^5, k \in \mathbb{N}$  are linearly

independent. This implies that basic Lie elements in  $D_k^1, D_k^5, k \in \mathbb{N}$  are also linearly independent, so they generate freely a free Lie subalgebra of  $\text{Im } \Phi_{01}^{\rightarrow}$ .

Remark. We have realized geometrically the fundamental group of  $\text{ML}_{\text{Spec } \mathcal{O}_{\mathbb{A}^1}^{(1-\mathbb{F}_3^*)}}(\mu_3)$ .  
 What about  $\pi_1(\text{ML}_{\text{Spec } \mathcal{O}_{\mathbb{A}^1}(\mu_2)})$ ?

Remark. If we look only at generators at degrees greater than 1, i.e.,

$$z_k^1 = Y_1 X^{k-1} + Y_2 X^{k-1} \quad z_k^5 = Y_3 X^{k-1} + Y_5 X^{k-1} \quad k=2,3,$$

then

$\{ \underbrace{z^m \dots z^j}_{\text{length } r} \} \equiv [ \dots z^k \ z ] \pmod{\text{vector subspace}}$   
 generated by Lie brackets with at least one  $Y_i$  with  $i$  odd and one  $Y_j$  with  $j$  even

So we can show much easier that  $z_k^1, z_k^5, k=2,3,\dots$  generate freely a free Lie subalgebra of  $\text{Im } \Phi_{01}^{\rightarrow}$ .

This can be generalized in the following way.

