

# Galois actions on fundamental groups.

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0. Let  $V := \mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$ . Let us fix a prime  $l$ . Let  $\pi_1(V_{\mathbb{Q}}; \vec{o}_1)$  be the maximal pro- $l$  quotient of the étale fundamental group of  $V_{\mathbb{Q}}$ . The tangential base point  $\vec{o}_1$  corresponds taking fiber over

$$\text{Spec } \Omega = \text{Spec} \left( \bigcup_{n=1}^{\infty} \overline{\mathbb{Q}}((z^{1/n})) \right) \longrightarrow \text{Spec } \overline{\mathbb{Q}}[z, \frac{1}{z}, \frac{1}{z^2}, \dots]$$

Laurent power series in  $z^{1/n}$

The Galois group  $G_{\mathbb{Q}}$  acts on  $\pi_1(V_{\mathbb{Q}}; \vec{o}_1)$ . This action was studied by Ihara, Deligne, Grothendieck. I shall follow mainly approach of Ihara. In this talk I would like to say something about action of Galois group on  $\pi_1(\mathbb{P}_{\mathbb{Q}}^1 \setminus (\{0, \infty\} \cup \mu_8); \vec{o}_1)$ .

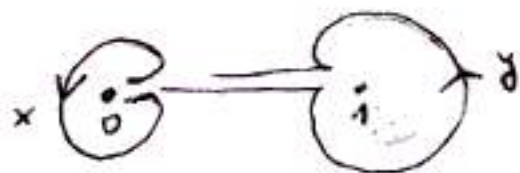
## 1. $l$ -adic Galois polylogarithms.

Let  $K$  be a number field and let  $z \in V(K)$ . Let  $\gamma$  be a path from  $\vec{o}_1$  to  $z$  on  $V_{\mathbb{Q}}$  (natural isomorphism of fiber functors over  $\vec{o}_1$  and over  $z$ ). For any  $\delta \in G_K$

we consider a loop

$$f_{\gamma}(\sigma) := \gamma^{-1} \cdot \sigma(\gamma) = \gamma^{-1} \circ (\sigma \circ \gamma \circ \sigma^{-1}) \in$$

$\pi_1(V_{\mathbb{K}}; \vec{o}_1)$  - free pro-l group on  $x$  and  $y$



$$f_{\gamma}(\sigma) \equiv x^{\alpha(\sigma)} \cdot y^{\beta(\sigma)} \pmod{\Gamma^2 \pi_1(V_{\mathbb{K}}; \vec{o}_1)}$$

We want to know coefficients  $\alpha(\sigma)$  and  $\beta(\sigma)$ .

Observe that

$$x: z^{1/n} \rightarrow \sum_{i=1}^n z^{i/n} \cdot z^{1/n}$$

$z$ -local parameters  
corresponding to  $\vec{o}_1$   
 $x$ : monodromy along  $x$

$$y: z^{1/n} \rightarrow z^{1/n}$$

Hence to calculate  $\alpha(\sigma)$  we need to act

$$\begin{aligned} f_{\gamma}(\sigma) &: z^{1/n} \xrightarrow{\sigma^{-1}} z^{1/n} \xrightarrow{\gamma} (z + z^{-2})^{1/n} = z^{1/n} \left(1 + \frac{z^{-2}}{z}\right)^{1/n} \\ &= z^{1/n} \left(1 + \binom{1/n}{1} \frac{z^{-2}}{z} + \dots\right) \xrightarrow{\sigma} \frac{\sigma(z^{1/n})}{z^{1/n}} \cdot z^{1/n} \xrightarrow{\gamma^{-1}} \sum_{i=1}^{K(z)(\sigma)} z^{i/n} \cdot z^{1/n} \end{aligned}$$

$$\text{hence } f_{\gamma}(\sigma) \equiv x^{K(z)(\sigma)} \cdot y^{K(1-z)(\sigma)} \pmod{\Gamma^2 \pi_1(V_{\mathbb{K}}; \vec{o}_1)}$$

Let

$k: \pi_1(V_{\mathbb{K}}; \vec{o}_1) \rightarrow \mathbb{Q}_2\langle\langle X, Y \rangle\rangle$  - formal power series in non-commutative variables  $X$  and  $Y$

