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The arithogeometric mean

By

Daniel R. Grayson*)

In this expository note we derive the arithogeometric mean method of Gauss. We assume the reader knows basic algebraic geometry and the theory of elliptic curves. We arrive at the required quadratic substitution with no guesswork, explain how to compute the periods of an elliptic curve, and how to compute the (inverse) elliptic integrals (elliptic functions). The algorithms described are suitable for machine computation to hundreds of digits of accuracy.

Begin with an elliptic curve E given by the equation $y^2 = x(x+r)(x+s)$. We assume that $(x+r)(x+s)$ has real coefficients, and that neither r nor s is real and nonnegative. Any real elliptic curve can be brought into this form. We let $L \subset \mathbb{C}$ be the lattice of periods of $\omega = \dfrac{dx}{2y}$, and let $\Phi\colon \mathbb{C}/L \xrightarrow{\approx} E(\mathbb{C})$ be the isomorphism with $\Phi^*\omega = dz$ (z = parameter on $\mathbb{C}$). We write $L = \mathbb{Z}\gamma + \mathbb{Z}\delta$ where $\gamma = 2\int\limits_0^\infty \omega$ is real. We know that $\Phi([0,\gamma])$ is a component of $E(\mathbb{R})$ containing $\Phi(0) = \infty$, so $\Phi(\frac{1}{2}\gamma) = (0,0)$. The other points of order two are $(-r, 0)$ and $(-s, 0)$, so we may assume $\Phi(\frac{1}{2}\delta) = (-r, 0)$ and $\Phi(\frac{1}{2}\delta + \frac{1}{2}\gamma) = (-s, 0)$ (by interchanging r and s, if necessary).

Let Λ be the lattice $\mathbb{Z}\gamma + \mathbb{Z}2\delta \subsetneqq \mathbb{C}$. It is invariant under complex conjugation, so it corresponds to a real elliptic curve. We find an equation $v^2 = u(u+R)(u+S)$ and an isomorphism $\Psi\colon \mathbb{C}/\Lambda \xrightarrow{\approx} F(\mathbb{C})$. The constant c in $\Psi^*\left(\dfrac{du}{2v}\right) = c\,dz$ may be assumed to be 1 by change of variable. We assume that R and S satisfy the same conditions that r and s did, so that $\Psi(0) = \infty$, $\Psi(\frac{1}{2}\gamma) = (0,0)$, $\Psi(\delta) = (-R, 0)$, and $\Psi(\delta + \frac{1}{2}\gamma) = (-S, 0)$.

The inclusion $\Lambda \subset L$ provides a 2-to-1 map (isogeny) $\varphi\colon F \longrightarrow E$ which makes the diagram

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \longrightarrow & \mathbb{C}/L \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ F(\mathbb{C}) & \xrightarrow[\varphi]{} & E(\mathbb{C}) \end{array}$$

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commute. We see that $\varphi^*\left(\frac{dx}{2y}\right) = \frac{du}{2v}$, $\varphi(0,0) = \varphi(-S,0) = (0,0)$, and $\varphi(\infty) = \varphi(-R,0) = \infty$. The explicit formulas which describe φ comprise the "Landen transformation"; we begin to calculate them now.

Projection on the x-axis gives an isomorphism $x: E/\sim \xrightarrow{\approx} \mathbb{P}^1$, where $\sim$ denotes the equivalence relation $P \sim -P$. Since φ is a group homomorphism, it induces a map $\psi: \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ on the quotients which is 2-to-1, and satisfies $\psi(0) = \psi(-S) = 0$, $\psi(\infty) = \psi(-R) = \infty$.

We write $x = \psi(u) = \frac{p(u)}{q(u)}$ where p and q are polynomials. Regarding u as unknown in the equation $xq(u) - p(u) = 0$, we see that $\max(\deg p, \deg q) = 2$. We must therefore have

$$x = \frac{ku(u+S)}{u+R}$$

where k is some nonzero constant. We deduce

$$ku^2 + (kS - x)u + (-xR) = 0,$$

which is a quadratic equation for u whose discriminant is

$$D(x) = x^2 + 2k(2R - S)x + k^2 S^2.$$

Now $D(x) = 0$ iff x is a branch point for ψ. If $x = x(\Phi(z))$, then $\Psi^{-1}(\psi^{-1}(\{x\})) = \{z, -z, z+\delta, -z+\delta\}$ modulo Λ, so x is a branch point iff $z = -z + \delta$ modulo Λ. Thus the branch points are given by taking $z = \frac{1}{2}\delta$ or $\frac{1}{2}\delta + \frac{1}{2}\gamma$, yielding $x = -r$ or $-s$. Equating $D(x)$ with $(x+r)(x+s)$ yields

$$\begin{aligned} r + s &= 2k(2R - S) \\ rs &= k^2 S^2. \end{aligned}$$

We compute

$$\begin{aligned} y^2 &= x(x+r)(x+s) = \\ &= \frac{ku(u+s)}{(u+R)}\left[\frac{ku(u+s)}{(u+R)} + r\right]\left[\frac{ku(u+s)}{(u+R)} + s\right] \\ &= k^3 v^2 (u+R)^{-4} (u^2 + 2Ru + RS)^2. \end{aligned}$$

Letting $c = \pm\sqrt{k}$ with appropriate sign, we find

$$y = c^3 v(u+R)^{-2}(u^2 + 2Ru + RS).$$

Now compute

$$\frac{du}{2v} = \varphi^*\left(\frac{dx}{2y}\right) = \frac{d\left(\frac{ku(u+S)}{u+R}\right)}{2c^3 v(u+R)^{-2}(u^2 + 2Ru + RS)} = c^{-1}\frac{du}{2v}$$

and conclude that $c = 1$ and $k = 1$.

We solve for R and S in terms of r and s and summarize what we've done:

$$
\begin{aligned}
&E: y^2 = x(x+r)(x+s) \quad \text{periods } \gamma, \delta\\
&\quad S = \sqrt{rs}\\
&\quad R = \tfrac{1}{2}(\tfrac{1}{2}(r+s)+S)\\
&F: v^2 = u(u+R)(u+S) \quad \text{periods } \gamma, 2\delta\\
&\quad \frac{du}{2v} = \frac{dx}{2y}\\
&\quad x = u(u+S)/(u+R)\\
&\quad y = v(u^2+2Ru+RS)/(u+R)^2.
\end{aligned}
$$

It turns out that we can avoid complex arithmetic here. Write $(x+r)(x+s) = x^2+bx+c$. Our assumptions about r and s imply that $c > 0$. We find

$$
\begin{aligned}
S &= +\sqrt{c}\\
R &= \tfrac{1}{2}(\tfrac{1}{2}b+S)
\end{aligned}
$$

and that R and S are real and positive.

Now we repeat this period-stretching procedure forever. Let $L_n = \mathbb{Z}\gamma + \mathbb{Z}2^n\delta$. We find the corresponding elliptic curve E_n (for $n \geqq 1$) as follows

$$
\begin{aligned}
&E_n\colon y_n^2 = x_n(x_n^2 + b_n x_n + c_n)\\
&\quad S_n = \sqrt{c_{n-1}}\\
&\quad R_n = \tfrac{1}{2}(\tfrac{1}{2}b_{n-1}+S_n)\\
&\quad b_n = R_n + S_n\\
&\quad c_n = R_n S_n\\
&\quad x_{n-1} = x_n(x_n+S_n)/(x_n+R_n)\\
&\quad y_{n-1} = y_n(x_n^2+2R_n x_n + R_n S_n)/(x_n+R_n)^2.
\end{aligned}
$$

(We set $R_0 = r$, $S_0 = s$, and notice $R_1 = R$, $S_1 = S$, $b_0 = b$, $c_0 = c$.)

To see the relation with the arithogeometric mean we set $A_n = \sqrt{R_n}$ and $B_n = \sqrt{S_n}$, and observe that $A_n = \frac{1}{2}(A_{n-1}+B_{n-1})$ and $B_n = \sqrt{A_{n-1}B_{n-1}}$. The limit $M = \lim A_n = \lim B_n$ exists and is the arithogeometric mean $\operatorname{agm}(A_0, B_0)$. We may compute it as $M = \sqrt{R_\infty} = \sqrt{S_\infty}$ where $R_\infty = \lim R_n$ and $S_\infty = \lim S_n$.

Introduce variables X and Y satisfying $Y^2 = X(X+R_\infty)(X+S_\infty) = X(X+M^2)^2$. Then we compute the real period

$$
\begin{aligned}
\gamma &= 2\int_0^\infty \frac{dx_0}{2y_0} = \int_0^\infty \frac{dx_0}{y_0} = \int_0^\infty \frac{dx_1}{y_1} = \cdots = \int_0^\infty \frac{dY}{Y} = \int_0^\infty \frac{dX}{\sqrt{X}(X+M^2)}\\
&= 2\int_0^\infty \frac{dt}{t^2+M^2} = \frac{2}{M}\int_0^\infty \frac{ds}{s^2+1} = \frac{\pi}{M}.
\end{aligned}
$$