

## EXACT SEQUENCES IN ALGEBRAIC K-THEORY

BY

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In this paper we provide a general framework for producing long exact sequences in algebraic  $K$ -theory.

Recall the space  $S\mathcal{M}$ , constructed in [8, p. 181] for any exact category  $\mathcal{M}$ , in terms of which the  $K$ -groups can be defined:  $K_i\mathcal{M} = \pi_{i+1}S\mathcal{M}$ . Suppose  $F: \mathcal{P} \rightarrow \mathcal{M}$  is an exact functor such that for each admissible monomorphism  $M \rightarrow N$  of  $\mathcal{M}$  there exists an admissible monomorphism  $M \rightarrow N'$  and a commutative diagram of admissible monomorphisms

$$\begin{array}{ccccccc} M & = & L_0 & \rightarrow & \cdots & \rightarrow & L_s \\ \downarrow & & & & & & \parallel \\ N \amalg_M N' & = & N_0 & \rightarrow & \cdots & \rightarrow & N_q \end{array}$$

in which each  $N_i/N_{i-1}$  or  $L_i/L_{i-1}$  is isomorphic to an object in the image of  $F$ . We prove that a certain square

$$\begin{array}{ccc} 0|SF & \rightarrow & S\mathcal{P} \\ \downarrow & & \downarrow SF \\ pt \sim 0|S\mathcal{M} & \rightarrow & S\mathcal{M} \end{array}$$

is homotopy cartesian, thereby yielding a long exact sequence

$$\cdots K_i\mathcal{M} \rightarrow \pi_i(0|SF) \rightarrow K_{i-1}\mathcal{P} \rightarrow K_{i-1}\mathcal{M} \cdots$$

The proof is based on the methods of [5].

It turns out that the criterion formulated above for  $F$  is often satisfied and the space  $0|SF$  can often be brought into homotopy equivalence with other interesting spaces. In this way we provide new proofs of the following theorems:

- (i) the cofinality theorem of Grayson and Waldhausen [4.1.1], [1];
- (ii) the dévissage theorem of Quillen [6, Theorem 4];
- (iii) the resolution theorem of Quillen [6, Theorem 3];
- (iv) the localization theorem for projection modules of Quillen [2], [3].

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The advantage of these new proofs is that they share substantial common ground, and the unique portions are thereby reduced in complexity.

In the final section, we apply the main theorem to produce an infinite sequence of definitions, for higher algebraic  $K$ -theory of an exact category, which were suggested by H. Gillet, and should enable one to define  $\lambda$ -operations on the higher  $K$ -groups of any exact category equipped with exterior powers.

I thank Henri Gillet for useful discussions.

### 1. The main theorem

We assume known all the terminology of [5].

We fix exact categories  $\mathcal{P}$  and  $\mathcal{M}$  and zero objects called 0 in each.

Let  $F: \mathcal{P} \rightarrow \mathcal{M}$  be an exact functor such that  $F(0) = 0$ , and consider the map

$$SF: S\mathcal{P} \rightarrow S\mathcal{M}.$$

For  $A \in \Delta$  and  $\bar{M} \in S\mathcal{M}(A)$  we consider the right fiber  $\bar{M}|SF$ ; a  $q$ -simplex  $W$  of it can be illustrated by the following diagram, which doesn't show the choices for the various quotients:

$$W = \left( \begin{array}{c} 0 = P_0 \rightarrow \dots \rightarrow P_q \\ \hline 0 = M_0 \rightarrow \dots \rightarrow M_a \rightarrow N_0 \rightarrow \dots \rightarrow N_q \end{array} \right)$$

Here the horizontal arrows are admissible monomorphisms,

$$A = [a] = \{0 < 1 < \dots < a\},$$

$$\bar{M} = (0 = M_0 \rightarrow \dots \rightarrow M_a),$$

the top row represents a  $q$ -simplex of  $S\mathcal{P}$ , the bottom row represents a  $q + a + 1$  simplex of  $S\mathcal{M}$ , and the double line represents the identity

$$\begin{array}{c} 0 = FP_0 \rightarrow \dots \rightarrow FP_q \\ \parallel \qquad \qquad \qquad \parallel \\ 0 = N_0/N_0 \rightarrow \dots \rightarrow N_q/N_0 \end{array}$$

between the right hand  $q$ -face of the bottom row and  $F$  applied to the top row. Here (and later) the notation  $N_i/N_j$  refers to the chosen quotient implicitly provided as part of the data constituting  $W$ .

In  $\mathcal{P}$  and  $\mathcal{M}$  we choose direct sums  $M \oplus N$  for all  $M$  and  $N$ , and choose pushouts  $L \amalg_M N$  for all pairs  $L \leftarrow M \rightarrow N$  of admissible monomorphisms.

We ensure the identities  $M \oplus 0 = M = 0 \oplus M$ ,  $L \amalg_M M = L = M \amalg_M L$ , and  $L \amalg_0 M = L \oplus M$ . For each admissible monomorphism  $M \rightarrow N$  we choose a cokernel  $\text{coker}(M \rightarrow N)$ , ensuring that  $\text{coker}(0 \rightarrow M) = M$  and  $\text{coker}(1_M) = 0$ .

It is an easy exercise to replace  $\mathcal{M}$  and  $\mathcal{P}$  by equivalent categories and provide direct sum operations which are strictly associative, have a strict identity, and for which  $F$  is strictly additive. It is a little bit harder (but still possible) to do the same thing for pushouts. Still, we will use some natural isomorphisms later that can't be made into identities, so we don't bother to make any of them into identities. Nevertheless, with our choice for direct sum, we get a map

$$+ : S\mathcal{M} \times S\mathcal{M} \rightarrow S\mathcal{M}$$

which makes  $S\mathcal{M}$  into a homotopy associative  $H$ -space, with 0 as (strict) identity.

We use the pushouts chosen above to define an operation

$$+ : \overline{M}|SF \times \overline{M}|SF \rightarrow \overline{M}|SF$$

by setting

$$W + W' := \left( \begin{array}{c} 0 = P_0 \oplus P'_0 \rightarrow \dots \rightarrow \underline{\underline{P_q \oplus P'_q}} \\ 0 = M_0 \rightarrow \dots \rightarrow M_a \rightarrow N_0 \amalg_{M_a} N'_0 \rightarrow \dots \rightarrow N_q \amalg_{M_a} N'_q \end{array} \right)$$

and specifying the undisplayed quotients as follows:

$$\begin{aligned} \frac{P_i \oplus P'_i}{P_j \oplus P'_j} &:= \frac{P_i}{P_j} \oplus \frac{P'_i}{P'_j}, & \frac{N_i \amalg_{M_a} N'_i}{N_j \amalg_{M_a} N'_j} &:= F\left(\frac{P_i \oplus P'_i}{P_j \oplus P'_j}\right), \\ \frac{N_i \amalg_{M_a} N'_i}{M_a} &:= \frac{N_i}{M_a} \oplus \frac{N'_i}{M_a}, \\ \frac{N_i \amalg_{M_a} N'_i}{M_j} &:= \text{coker}(M_j \rightarrow N_i \amalg_{M_a} N'_i) \quad \text{for } 1 \leq j < a. \end{aligned}$$

We see that  $+$  is a simplicial map. The base-change map  $\overline{M}|SF \rightarrow 0|SF$  which factors out  $M_a$  preserves  $+$  strictly. The projection map  $\pi: \overline{M}|SF \rightarrow S\mathcal{P}$  preserves  $+$  strictly. As in [5], we use natural transformations to see that  $\overline{M}|SF$  is a homotopy associative and homotopy commutative  $H$ -space. This gives  $\pi_0(\overline{M}|SF)$  an operation  $+$  and a homotopy identity element 0 which

makes it into a (commutative) monoid. The simplex which represents 0 is:

$$0 := \left( \begin{array}{ccccccc} & & & & & & 0 = P_0 \\ & & & & & & \underline{\underline{\phantom{M_a}}} \\ 0 = M_0 & \twoheadrightarrow & \dots & \twoheadrightarrow & M_a & \xrightarrow{1} & M_a \end{array} \right)$$

**THEOREM C'.** *Suppose for all  $A \in \Delta$  and all  $\bar{M} \in S\mathcal{M}(A)$  that the monoid  $\pi_0(\bar{M}|SF)$  is a group. Then the square*

$$\begin{array}{ccc} 0|SF & \rightarrow & S\mathcal{P} \\ \downarrow & & \downarrow \\ pt \sim 0|S\mathcal{M} & \rightarrow & S\mathcal{M} \end{array}$$

is homotopy cartesian.

*Proof.* By Theorem B' of [5], we need only show for any  $f: A' \rightarrow A$  in  $\Delta$ , that the base change map  $f^*: \bar{M}|SF \rightarrow f^*\bar{M}|SF$  is a homotopy equivalence.

Let  $g: [0] \rightarrow A'$  be any map in  $\Delta$ . It suffices to show that  $g^*f^* = (fg)^*$  and  $g^*$  are homotopy equivalences. Both  $fg$  and  $g$  have  $[0]$  as source, so replacing  $f$  by either of them, we may assume that  $f$  is the map  $f_i: [0] \rightarrow A$ , defined for  $i \in A$  by  $f_i(0) = i$ . Thus  $f_i^*$  is a map  $\bar{M}|SF \rightarrow 0|SF$  (because 0 is the only vertex of  $S\mathcal{M}$ ).

We define a map  $H: 0|SF \rightarrow \bar{M}|SF$  by direct sum with  $\bar{M}$ , i.e., by

$$\begin{array}{c} \left( \begin{array}{ccccccc} 0 = P_0 & \twoheadrightarrow & \dots & \twoheadrightarrow & P_q \\ \underline{\underline{\phantom{N_0}}} & & & & \underline{\underline{\phantom{N_q}}} \\ 0 & \twoheadrightarrow & N_0 & \twoheadrightarrow & \dots & \twoheadrightarrow & N_q \end{array} \right) \\ \downarrow H \\ \left( \begin{array}{ccccccc} 0 = P_0 & \twoheadrightarrow & \dots & \twoheadrightarrow & P_q \\ \underline{\underline{\phantom{M_a \oplus N_0}}} & & & & \underline{\underline{\phantom{M_a \oplus N_q}}} \\ 0 = M_0 & \twoheadrightarrow & \dots & \twoheadrightarrow & M_a & \twoheadrightarrow & M_a \oplus N_0 & \twoheadrightarrow & \dots & \twoheadrightarrow & M_a \oplus N_q \end{array} \right) \end{array}$$

The undisplayed quotients here are defined as follows.

$$\frac{M_a \oplus N_j}{M_a \oplus N_k} := \frac{N_j}{N_k} \quad (= F(P_j/P_k)),$$

$$\frac{M_a \oplus N_j}{M_a} := N_j, \quad \frac{M_a \oplus N_j}{M_i} := \frac{M_a}{M_i} \oplus N_j.$$

The map  $f_i^* \circ H: 0|SF \rightarrow 0|SF$  is depicted thus:

$$\begin{array}{c} \left( \begin{array}{ccccccc} 0 = P_0 & \rightarrow & \cdots & \rightarrow & P_q \\ 0 & \rightarrow & N_0 & \rightarrow & \cdots & \rightarrow & N_q \end{array} \right) \\ \downarrow f_i^* \circ H \\ \left( \begin{array}{ccccccc} 0 = P_0 & \rightarrow & \cdots & \rightarrow & P_q \\ \hline 0 & \rightarrow & \frac{M_a}{M_i} \oplus N_0 & \rightarrow & \cdots & \rightarrow & \frac{M_a}{M_i} \oplus N_q \end{array} \right) \end{array}$$

for  $0 \leq i \leq a$ , and  $f_a^* \circ H$  is the identity. Thus the map  $f_i^* \circ H$  is the translation on the  $H$ -space  $0|SF$  obtained by adding the vertex

$$\left( \begin{array}{c} 0 \\ \hline 0 \rightarrow \frac{M_a}{M_i} \end{array} \right);$$

since its homotopy class is in the image of the monoid map  $\pi_0(0|SF) \rightarrow [0|SF, 0|SF]$ , and  $\pi_0(0|SF)$  is a group, we see that  $f_i^* \circ H$  is a homotopy equivalence. (Here  $[-, -]$  denotes homotopy classes of maps.)

If we show  $H$  is a homotopy equivalence, then it will follow that  $f_i^*$  is a homotopy equivalence for each  $i$ . In order to show that  $H$  is a homotopy equivalence, it is enough to show that  $H \circ f_a^*$  is homotopic to the identity map 1 of  $\overline{M}|SF$ . The desired homotopy results from the natural isomorphism

$$1 + H \circ f_a^* \cong 1 + 1,$$

which in turn amounts to the natural isomorphism

$$N_j \amalg_{M_a} \left( M_a \oplus \frac{N_j}{M_a} \right) \cong N_j \amalg_{M_a} N_j. \quad \text{Q.E.D.}$$

## 2. Interpretation

In this section we interpret the hypothesis of Theorem  $C'$ , and say that  $F$  is *dominant* if the hypothesis is satisfied. The method is to describe  $\pi_0(\overline{M}|SF)$  in terms of generators and relations.

An admissible monomorphism of  $\mathcal{M}$  whose cokernel is isomorphic to an object in the image of  $F$  will be called an  $F$ -mono.

We start with the vertices as generators, and all 1-simplices as relations. If two vertices are isomorphic, then they are joined by a 1-simplex. Thus we may forget about the choices for quotients, and take as generators all admissible monomorphisms  $(M_a \twoheadrightarrow N)$  (here  $N$  being variable, and  $M_a$ , as part of  $\overline{M}$ , being fixed). We take as *elementary* relations the following:

$$(M_a \twoheadrightarrow N_0) \sim (M_a \twoheadrightarrow N_1)$$

if there is an  $F$ -mono  $i: N_0 \twoheadrightarrow N_1$  such that

$$\begin{array}{ccc} M_a \twoheadrightarrow N_0 & & \\ \parallel & \Downarrow & \\ M_a \twoheadrightarrow N_1 & & \end{array}$$

commutes. The operation  $+$  is expressed by

$$(M_a \twoheadrightarrow N_0) + (M_a \twoheadrightarrow N_1) := (M_a \twoheadrightarrow N_0 \amalg_{M_a} N_1),$$

and respects the relations.

If  $L \leftarrow N \twoheadrightarrow M$  are  $F$ -monos, then so are all the arrows in the corresponding pushout diagram. Thus we may convert any chain of elementary equivalences into a chain where the arrows of the same sense are grouped together:

$$\begin{array}{ccccccccccc} M_a = M_a = \cdots = M_a = M_a = \cdots = M_a = M_a & & & & & & & & & & \\ \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & & & \\ N_0 \twoheadrightarrow N_1 \twoheadrightarrow \cdots \twoheadrightarrow N_i = N'_j \leftarrow \cdots \leftarrow N'_1 \leftarrow N'_0. & & & & & & & & & & \end{array}$$

Thus we arrive at the following theorem.

**THEOREM 2.1.** *The exact functor  $F$  is dominant (and thus Theorem C' applies to it) if and only if for each admissible monomorphism  $M \twoheadrightarrow N$  of  $\mathcal{M}$  there exists an admissible monomorphism  $M \twoheadrightarrow N'$  and a commutative diagram*

$$\begin{array}{ccccccc} M & = & L_0 \twoheadrightarrow & \cdots & \twoheadrightarrow & L_s & \\ \Downarrow & & & & & \parallel & \\ N \amalg_M N' & = & N_0 \twoheadrightarrow & \cdots & \twoheadrightarrow & N_q & \end{array}$$

in which the horizontal arrows are  $F$ -monos.

*Remark.* Theorem C' is asymmetric, in the sense that it may apply to  $F^{\text{op}}$  when it doesn't apply to  $F$ . A similar theorem which is symmetric is desirable.

*Remark.* Edgewise subdivision of the square in Theorem C' gives a homotopy cartesian square

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ Q\mathcal{P} & \longrightarrow & Q\mathcal{M} \end{array}$$

of categories, where  $E$  is the extension construction of  $\mathcal{M}$ , and  $E'$  is the pullback. This fact can be deduced (as pointed out by Swan [7, p. 152]) from the proof in [2], [3] that the square

$$\begin{array}{ccc} S^{-1}E' & \rightarrow & S^{-1}E \\ \downarrow & & \downarrow \\ Q\mathcal{P} & \rightarrow & Q\mathcal{M} \end{array}$$

is homotopy cartesian, but that proof depends on the presence of the localization  $S^{-1}$ , for which one needs the additional hypothesis that every object in  $\mathcal{M}$  have projective dimension  $\leq 1$  within  $\mathcal{M}$ .

Define  $\text{im } F$  to be the set of those objects of  $\mathcal{M}$  isomorphic to  $F(P)$  for some  $P \in \text{obj } \mathcal{P}$ .

**COROLLARY 2.2.** *If  $\text{im } F = \text{obj } \mathcal{M}$  (i.e.,  $F$  is surjective on isomorphism classes of objects) then  $F$  is dominant.*

Call  $\text{im } F$  *cofinal* in  $\text{obj } \mathcal{M}$  if for any  $T \in \text{obj } \mathcal{M}$  there is a  $T' \in \text{obj } \mathcal{M}$  with  $T \oplus T' \in \text{im } F$ .

**COROLLARY 2.3.** *If  $\text{im } F$  is cofinal in  $\text{obj } \mathcal{M}$ , then  $F$  is dominant.*

*Proof.* Given  $M \twoheadrightarrow N$  as in the theorem, we find  $T'$  so

$$\frac{N}{M} \oplus T' \in \text{im } F.$$

Set  $N' = M \oplus T'$ , so  $N \amalg_M N' = N \oplus T'$ . The inclusion  $M \twoheadrightarrow N \amalg_M N'$  is itself an  $F$ -mono. Q.E.D.

Call  $\text{im } F$  *closed under direct summand* if  $M \oplus N \in \text{im } F$  implies  $M \in \text{im } F$ .

Call  $\text{im } F$  *closed under extension* if exactness of

$$0 \rightarrow F(P') \rightarrow M \rightarrow F(P'') \rightarrow 0$$

in  $\mathcal{M}$  implies  $M \in \text{im } F$ .

Call  $\text{im } F$  *closed under cokernel* if exactness of

$$0 \rightarrow F(P') \rightarrow F(P) \rightarrow M'' \rightarrow 0$$

in  $\mathcal{M}$  implies  $M'' \in \text{im } F$ . Similarly for *closed under kernel*.

**COROLLARY 2.4.** *If  $\text{im } F$  is closed under direct summand, extension, and cokernel, then  $F$  is dominant if and only if every  $M \in \text{obj } \mathcal{M}$  fits into an exact sequence  $0 \rightarrow M \rightarrow F(P) \rightarrow F(P'') \rightarrow 0$ .*

*Proof.* Since  $\text{im } F$  is closed under extension, the composition of two  $F$ -monos is again an  $F$ -mono (to see this, apply the ker-coker exact sequence for a composition).

Assuming  $F$  is dominant, we consider the mono  $0 \rightarrow M$  and find  $M' \in \text{obj } \mathcal{M}$  and  $F$ -monos  $0 \rightarrow L \leftarrow M \oplus M'$ . Thus we have an exact sequence

$$0 \rightarrow M \oplus M' \xrightarrow{(f, g)} F(P) \rightarrow F(P'') \rightarrow 0.$$

Consider the ker-coker exact sequence for the composition

$$M \oplus M' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} F(P) \oplus F(P) \xrightarrow{+} F(P);$$

it reduces to

$$0 \rightarrow F(P) \rightarrow \frac{F(P)}{M} \oplus \frac{F(P)}{M'} \rightarrow F(P'') \rightarrow 0,$$

and allows us to deduce that  $F(P)/M \in \text{im } F$ , which is what we wanted.

Now we try to prove  $F$  is dominant, assuming the other hypothesis. Starting with  $M \rightarrow N$  as in the theorem we find exact sequences

$$0 \rightarrow M \rightarrow F(P) \rightarrow F(P'') \rightarrow 0, \quad 0 \rightarrow N \rightarrow F(Q) \rightarrow F(Q'') \rightarrow 0$$

and

$$0 \rightarrow M \rightarrow F(Q) \rightarrow L \rightarrow 0.$$

Now form the pushout  $K$  as in the diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & M & \rightarrow & F(P) & \rightarrow & F(P'') \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & F(Q) & \rightarrow & K & \rightarrow & F(P'') \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & L & = & L & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Exactness of the middle row shows  $K \in \text{im } F$ , and exactness of the middle column shows  $L \in \text{im } F$ . The commutative diagram

$$\begin{array}{ccc} M & \twoheadrightarrow & F(Q) \\ \downarrow & & \parallel \\ N & \twoheadrightarrow & F(Q) \end{array}$$

in which the horizontal arrows are  $F$ -monos shows  $F$  is dominant.

**COROLLARY 2.5.** *If  $\text{im } F$  is closed under extension and cokernel, and every  $M \in \text{obj } \mathcal{M}$  fits into an exact sequence  $0 \rightarrow M \rightarrow F(P) \rightarrow F(P'') \rightarrow 0$ , then  $F$  is dominant.*

*Proof.* Same as half of the proof of 2.4.

Q.E.D.

Call  $\text{im } F$  *closed under subobject* if  $0 \rightarrow M' \rightarrow F(P) \rightarrow M'' \rightarrow 0$  being exact implies  $M' \in \text{im } F$ . Similarly for *closed under quotient object*.

**COROLLARY 2.6.** *Suppose  $\mathcal{M}$  is an abelian category, and  $\text{im } F$  is closed under subobject. Then  $F$  is dominant if and only if every object of  $\mathcal{M}$  has a finite filtration whose consecutive quotients are in  $\text{im } F$ .*

*Proof.* First assume the condition about filtrations. Then given  $M \twoheadrightarrow N$  an admissible mono as in 2.1, we may find such a filtration for  $N/M$ , which expresses  $M \twoheadrightarrow N$  as a composition of  $F$ -monos, showing  $F$  is dominant.

Now assume  $F$  is dominant. Then by 2.1, for each  $M \in \mathcal{M}$  we may find an  $M'$  and  $F$ -monos  $M \oplus M' = N_0 \twoheadrightarrow \cdots \twoheadrightarrow N_q = L_s \leftarrow \cdots \leftarrow L_0 = 0$ . We have

$$\frac{L_i \cap M}{L_{i-1} \cap M} \subset \frac{L_i}{L_{i-1}} \in \text{im } F,$$

so the consecutive quotients of the filtration

$$0 = L_0 \cap M \subset \cdots \subset L_q \cap M = M$$

are in  $\text{im } F$ .

Q.E.D.

### 3. Application

Call an exact subcategory  $\mathcal{P}$  of an exact category  $\mathcal{M}$  *closed under exact sequences* if any exact sequence of  $\mathcal{M}$  whose members are objects of  $\mathcal{P}$ , is also an exact sequence of  $\mathcal{P}$ .

**THEOREM 3.1.** *Assume  $\mathcal{P} \subset \mathcal{M}$  is a full exact subcategory closed under exact sequences and under extension. Let  $F: \mathcal{P} \rightarrow \mathcal{M}$  denote the inclusion. Then  $0|SF$  is homotopy equivalent to the subcategory  $\mathcal{C}_F$  of  $\mathcal{M}$  whose arrows are all the admissible monomorphisms of  $\mathcal{M}$  with cokernel isomorphic to an object of  $\mathcal{P}$ .*

*Proof.* The set of arrows mentioned is closed under composition because  $\mathcal{P}$  is closed under extension in  $\mathcal{M}$ , and thus  $\mathcal{C}_F$  is a category. The obvious map  $g: 0|SF \rightarrow \mathcal{C}_F$  depicted by

$$\begin{array}{c} \left( \begin{array}{c} 0 = P_0 \rightarrow \cdots \rightarrow P_q \\ \hline 0 \rightarrow M_0 \rightarrow \cdots \rightarrow M_q \end{array} \right) \\ \downarrow g \\ (M_0 \rightarrow \cdots \rightarrow M_q) \end{array}$$

simply forgets the choices for the quotients  $M_j/M_i$ . Choosing cokernels in  $\mathcal{P}$  for all  $F$ -monos gives a map  $h: \mathcal{C}_F \rightarrow 0|SF$  the other way, with  $g \circ h = 1$ . The natural isomorphism  $h \circ g \cong 1$  provides a homotopy  $h \circ g \sim 1$ . Q.E.D.

#### 4. The resolution theorem

Define  $i\mathcal{M}$  to be the subcategory of  $\mathcal{M}$  whose arrows are the admissible monomorphisms of  $\mathcal{M}$ . It has 0 as initial object, so is contractible.

**THEOREM 4.1.** *Assume  $\mathcal{P} \subset \mathcal{M}$  is a full exact subcategory closed under exact sequences, extension, and cokernel. Assume that any  $M \in \mathcal{M}$  has a resolution*

$$0 \rightarrow M \rightarrow P \rightarrow P'' \rightarrow 0$$

*with  $P, P'' \in \mathcal{P}$ . Then the map  $S\mathcal{P} \rightarrow S\mathcal{M}$  is a homotopy equivalence.*

*Proof.* By 3.1 and 2.5, it is enough to show that  $\mathcal{C}_F$  is contractible, where  $F: \mathcal{P} \rightarrow \mathcal{M}$  is the inclusion. It is enough to show that the inclusion map  $G: i\mathcal{P} \rightarrow \mathcal{C}_F$  is a homotopy equivalence. For this we will use Theorem A of [6], so for any  $M \in \mathcal{M}$  we consider the fiber  $M|G$  (or  $M/G$  in the notation of Quillen), and show  $M|G$  is contractible.

Choose a resolution  $0 \rightarrow M \rightarrow P_0 \rightarrow P_0'' \rightarrow 0$  with  $P_0', P_0'' \in \mathcal{P}$ ; the mono  $M \rightarrow P_0$  is an object of  $M|G$ . Suppose  $M \rightarrow P$  is any other object of  $M|G$ .

Pushout gives a diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & M & \rightarrow & P & \rightarrow & M'' \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \rightarrow & P_0 & \rightarrow & P_0 \amalg_M P & \rightarrow & M'' \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & P_0'' = & & P_0'' & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

from which we see that  $P_0 \amalg_M P$  and hence  $M''$  are in  $\mathcal{P}$ . The arrows

$$(M \twoheadrightarrow P) \rightarrow (M \twoheadrightarrow P_0 \amalg_M P) \leftarrow (M \twoheadrightarrow P_0)$$

in  $M|G$  constitute natural transformations which show that the identity map on  $M|G$  is homotopic to a constant map. Q.E.D.

### 5. The Dévissage theorem

**THEOREM 5.1 [6].** *Suppose  $\mathcal{M}$  is an abelian category, and  $\mathcal{P} \subset \mathcal{M}$  is a full (abelian) subcategory closed under direct sum, subobject, and quotient object. If every object of  $\mathcal{M}$  has a finite filtration whose consecutive quotients are in  $\mathcal{P}$ , then the map  $S\mathcal{P} \rightarrow S\mathcal{M}$  is a homotopy equivalence.*

*Proof.* Let  $F: \mathcal{P} \rightarrow \mathcal{M}$  be the inclusion. By 2.6 it is enough to show that  $0|SF$  is contractible. Since  $\mathcal{P}$  is a subcategory of  $\mathcal{M}$ , we see (by the same argument as in the proof of 3.1) that  $0|SF$  is homotopy equivalent to the simplicial set  $\mathcal{D}_F$ , whose  $q$ -simplices are all chains  $M_0 \twoheadrightarrow \cdots \twoheadrightarrow M_q$  of admissible monomorphisms in  $\mathcal{M}$ , such that  $M_q/M_0$  is in  $\mathcal{P}$ .

It will suffice to show that the inclusion map  $G: \mathcal{D}_F \rightarrow i\mathcal{M}$  is a homotopy equivalence. For this we will use Theorem A' of [5], so for any  $M \in \mathcal{M}$  we still show the fiber  $G|M$  is contractible. Associating an admissible monomorphism  $N \twoheadrightarrow M$  with the corresponding subobject of  $M$  gives a map  $g: G|M \rightarrow \mathcal{B}_M$ , where  $\mathcal{B}_M$  is the simplicial set whose  $q$ -simplices are all chains  $N_0 \subset \cdots \subset N_q \subset M$  of subobjects of  $M$  such that  $N_q/N_0$  is in  $\mathcal{P}$ . The map  $g$  is a homotopy equivalence, as it has an inverse up to natural isomorphism.

Now choose a filtration  $0 = M_0 \subset \cdots \subset M_s = M$  of  $M$  such that  $M_i/M_{i-1} \in \mathcal{P}$  for all  $i$ . Define a map  $F_i: \mathcal{B}_M \rightarrow \mathcal{B}_M$  by

$$F_i: (N_0 \subset \cdots \subset N_q \subset M) \mapsto (N_0 + M_i \subset \cdots \subset N_q + M_i \subset M).$$

Since  $N_q + M_i/N_0 + M_i$  is a quotient of  $N_q/N_0$ , it is in  $\mathcal{P}$  and the latter chain is a simplex of  $\mathcal{B}_M$ .

The inclusions  $N_j + M_{i-1} \subset N_j + M_i$  give a homotopy from  $F_{i-1}$  to  $F_i$  because

$$\frac{N_q + M_i}{N_0 + M_{i-1}}$$

is a quotient of

$$\frac{N_q}{N_0} \oplus \frac{M_i}{M_{i-1}},$$

and thus is in  $\mathcal{P}$ .

Since  $F_0 = 1_{\mathcal{B}_M}$  and  $F_s$  is constant, we see that  $\mathcal{B}_M$  is contractible. Q.E.D.

### 6. Cofinality

**THEOREM 6.1** [4, 1.1], [1]. *Assume  $\mathcal{P} \subset \mathcal{M}$  is a full exact subcategory closed under exact sequences and extension. If  $\text{obj } \mathcal{P}$  is cofinal in  $\text{obj } \mathcal{M}$ , then  $K_i \mathcal{P} \rightarrow K_i \mathcal{M}$  is an isomorphism for  $i \geq 1$ , and a monomorphism for  $i = 0$ .*

*Proof.* By 3.1, 2.3, and Theorem C', we see that it is enough to show that  $\pi_i(\mathcal{C}_F) = 0$  for  $i \geq 1$ , where  $F$  denotes the inclusion  $\mathcal{P} \rightarrow \mathcal{M}$ . Let  $\mathcal{B}$  be the connected component of 0 in  $\mathcal{C}_F$ .

If  $M$  is an object of  $\mathcal{B}$ , then as an object of  $\mathcal{M}$ , its class in  $K_0 \mathcal{M}$  lies in the image of the map  $K_0 \mathcal{P} \rightarrow K_0 \mathcal{M}$ . By an elementary argument from [4, Section 1] there exists  $P \in \mathcal{P}$  so  $M \oplus P \in \mathcal{P}$ .

Now suppose  $\mathcal{A} \subset \mathcal{B}$  is any sub-simplicial set with a finite number of objects. By summing the  $P$ 's found above for each  $M \in \mathcal{A}$ , we may find a  $P \in \mathcal{P}$  so that  $M \oplus P \in \mathcal{P}$  for all  $M \in \mathcal{A}$ . This  $P$  can be used to construct a null-homotopy for  $\mathcal{A} \hookrightarrow \mathcal{B}$  from the following pair of simplicial homotopies:

$$\begin{array}{c} (M_0 \twoheadrightarrow \dots \twoheadrightarrow M_q) \\ \downarrow \\ (M_0 \oplus P \twoheadrightarrow \dots \twoheadrightarrow M_q \oplus P) \\ \uparrow \\ (0 \twoheadrightarrow \dots \twoheadrightarrow 0) \end{array}$$

Checking that these maps are actually simplicial homotopies amounts to the assertion that  $(M_q \oplus P)/M_0$  and  $M_q \oplus P$  are in  $\mathcal{P}$ , which is what we have arranged above.

It follows now that  $\pi_i(\mathcal{C}_F) = \pi_i(\mathcal{B}) = 0$  for  $i \geq 1$ .

Q.E.D.

