HIGHER ALGEBRAIC K-THEORY : II

[after Daniel Quillen]

by Daniel Grayson

The purpose of this paper is to prove three theorems announced in Higher Algebraic K-theory : I by Quillen.

The first theorem says that the two definitions of $K_i(R)$ offered by Quillen agree. The notion of monoidal category allows the construction of a fibration which shows that $K_0R \times BGl(R)^+$ is the loop space of $QP(R)$.

The localization theorem concerns the localization of a ring, $R \longrightarrow S^{-1}R$, and identifies the third term in the long exact sequence. It was first proved for K_0 and K_1 by Bass. Gersten gave a proof for the higher K_i and conjectured the non-affine version proved here.

The fundamental theorem describes extensions $R \longrightarrow R[t]$ and $R \longrightarrow R[t,t^{-1}]$, and was first proved by Bass. The proof here involves an application of the localization theorem to the projective line.

I thank Daniel Quillen for communicating to me his proofs of these theorems and for his helpful explanations of them.

Monoidal Categories and Localization

Suppose S is an abelian monoid which acts on a set X. S acts invertibly on X if each translation

$$X \longrightarrow X$$
$$x \longmapsto sx \quad ,$$

for $s \in S$, is a bijection. Define $S^{-1}X$ to be the quotient $S{\times}X/S$, where S acts on each factor of the product. Let S act on $S^{-1}X$ by $t\cdot(s,x) = (s,tx)$, and define $X \longrightarrow S^{-1}X$ to be $x \longmapsto (1,x)$. Then the translation $(s,x) \longmapsto (s,tx)$ has an inverse given by $(s,x) \longmapsto (ts,x)$, so S acts invertibly on $S^{-1}X$. The arrow $X \longrightarrow S^{-1}X$ respects the S-action, and is a universal arrow from X to a set upon which S acts invertibly.

If S is abelian, then the set $S^{-1}S$ is a group under the multiplication $(s,t)\cdot(u,v) = (su,tv)$. The map $S \longrightarrow S^{-1}S$ is a homomorphism, and is a universal arrow from S to a group.

This notion of localization is placed in the context of categories as follows.

Def: A monoidal category S is a category S with an operation $+ : S{\times}S \longrightarrow S$ and an object 0. There are natural isomorphisms $A+(B+C) \cong (A+B)+C$, $0+A \cong A$, $A+0 \cong A$. The following diagrams must commute :

$$\begin{array}{ccccc} A+(B+(C+D)) & \cong & (A+B)+(C+D) & \cong & ((A+B)+C)+D \\ \wr\!\shortparallel & & & & \wr\!\shortparallel \\ A+((B+C)+D) & & \cong & & (A+(B+C))+D \end{array} ,$$

$$\begin{array}{ccc} A+(0+C) & \cong & (A+0)+C \\ \wr\!\shortparallel & & \wr\!\shortparallel \\ A+C & = & A+C \end{array} . \quad \text{(see MacLane)}$$

Def: A left action of a monoidal category S on a category X is a functor $+ : S{\times}X \longrightarrow X$ with natural isomorphisms $A+(B+F) \cong (A+B)+F$ and $0+F \cong F$, where $A,B \in S$ and $F \in X$. Diagrams analogous to those above must commute.

Def: A monoidal functor is a functor $S \xrightarrow{f} T$ where S and T are monoidal categories, equipped with natural isomorphisms $f(A+B) \cong fA + fB$ and $f0 \cong 0$. The following diagrams must commute :

$$\begin{array}{ccccc} f((A+B)+C) & \cong & f(A+B)+fC & \cong & (fA+fB)+fC \\ \wr\!\shortparallel & & & & \wr\!\shortparallel \\ f(A+(B+C)) & \cong & fA+f(B+C) & \cong & fA+(fB+fC) \end{array}$$

$$\begin{array}{ccc} f(0+A) & \cong & f0 + fA \\ \wr\!\shortparallel & & \wr\!\shortparallel \\ fA & \cong & 0 + fA \end{array} \qquad \begin{array}{ccc} f(A+0) & \cong & fA + f0 \\ \wr\!\shortparallel & & \wr\!\shortparallel \\ fA & \cong & fA + 0 \end{array} .$$

Def: A functor $g : X \longrightarrow Y$ of categories with S-action, preserves the action if there is a natural isomorphism $A+gF \cong g(A+F)$, and

$$\begin{array}{ccccc} (A+B)+gF & \cong & g((A+B)+F) & \cong & g(A+(B+F)) \\ \wr\!\shortparallel & & & & \wr\!\shortparallel \\ A+(B+gF) & & \cong & & A+g(B+F) \end{array} \qquad \text{and}$$

$$\begin{array}{ccc} 0 + gF & \cong & g(0+F) \\ \wr\| & & \wr\| \\ gF & = & gF \end{array} \quad \text{commute.}$$

The commutativity of these diagrams yields the commutativity of every diagram which should commute. For details, see (MacLane). This commutativity assures us that the constructions to be made will satisfy the axioms for category or functor.

For details and notation about topological notions applied to categories, and for the theorem about constructing fibrations, the reader should refer to (Quillen).

<u>Def</u>: If S is a monoidal category which acts on a category X, then S acts <u>invertibly</u> on X if each translation

$$\begin{array}{ccc} X & \longrightarrow & X \\ F & \longmapsto & A+F \end{array}$$

is a homotopy equivalence.

<u>Def</u>: If S acts on X, the category $\langle S,X\rangle$ has the same objects as X. An arrow is represented by an isomorphism class of tuples $(F,G,A,A+F\longrightarrow G)$ with $A \in S$ and F,G in X. This arrow is an arrow from F to G. An isomorphism of tuples is an isomorphism $A \cong A'$ which makes

$$\begin{array}{ccc} A+F & \cong & A'+F \\ & \searrow \ \swarrow & \\ & G & \end{array} \quad \text{commute.}$$

<u>Def</u>: The category $S^{-1}X$ is $\langle S,S\times X\rangle$, where S acts on both factors of the product. The action of S on $S^{-1}X$ is given by $A + (B,F) = (B,A+F)$, if S is commutative up to natural isomorphism.

Notice that S acts invertibly on $S^{-1}X$. The translation $(B,F) \mapsto (B,A+F)$ has homotopy inverse $(B,F) \mapsto (A+B,F)$, in light of the natural transformation $(B,F)\longrightarrow(A+B,A+F)$.

<u>Note</u>: If every arrow is an isomorphism in S, then $\langle S,S\rangle$ has initial object 0 and is contractible. We now make the blanket assumption for the rest of the paper that this condition holds. In practice, S is usually the groupoid of isomorphisms in an exact category.