

Reduction theory using semistability, II

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In this paper we extend the result of [G] to include the case of an arithmetic subgroup Γ of a semisimple algebraic group G . We represent the symmetric space $X = G/K$ (where $K =$ a maximal compact subgroup of G) as a certain space of inner products H on the Lie algebra \mathfrak{g} of G , namely, those inner products that come from the Killing form via a Cartan involution.

If we assume that Γ is the stabilizer of a lattice $\mathfrak{g}_Z \subset \mathfrak{g}$, then we may use the notions of semistability and of Harder–Narasimhan canonical filtration to study the action of Γ on X . Our main result is the explicit construction of a closed submanifold (with boundary) in X (of codimension zero) which is contractible, compact modulo Γ , and has boundary Γ -homotopy equivalent to the Tits building of G . Thus this subspace of X provides an alternate route toward the theorems of Borel–Serre [BS] about the homological properties of Γ .

The proof in [G] of contractibility for the intersections of the neighborhoods of the cusps [G, 7.18(c), 7.7] had the side effect of proving something already known, namely contractibility of the space X . In section 4 we use a simpler technique that makes use of the contractibility of X .

We make essential use of the assumption that G is semisimple; it would have been nice to avoid it altogether, the way Borel–Serre do.

The idea of formulating the results of reduction theory in terms of functions which measure the distance to the cusps; here and in [G], is due to Harder [Ha1, Ha2]. Indeed, his function $\pi(P, \mathcal{R})$ of [Ha1, 2.4, p. 54] is presumably closely related to our $\text{vol}(L \cap W^1)$, where W^1 is the nilpotent radical of a parabolic subalgebra of \mathfrak{g} . In a sense, we answer the question in the last sentence of [Ha1], hopefully by a method as direct and transparent as the one anticipated there by Harder, thereby completing the program he laid out.

Motivation for doing reduction theory via semistability of Lie algebra bundles comes from the paper of Atiyah and Bott [AB], where the canonical filtration for a principal G -bundle on a Riemann surface is introduced.

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1. Inner products on a Lie algebra

Let G be a connected semisimple algebraic group defined over \mathbb{Q} . Since we will be dealing with the adjoint representation of G , we will replace G by $G' = G/Z(G)$, where Z = "center of." Since G is semisimple, $Z(G)$ is finite, and G' is also a semisimple linear algebraic group; its adjoint representation is now faithful, so $Z(G') = 1$ [W, 3.50]. Any arithmetic subgroup $\Gamma \subset G(\mathbb{R})$ has a congruence subgroup $\Gamma' \subset \Gamma$ of finite index which maps injectively to $G'(\mathbb{R})$; its image in G' is an arithmetic subgroup of G' [B, 7.13(2)]. Since all theorems we intend to prove about Γ are (equivalent to a theorem) of the form " Γ has a subgroup Γ' of finite index such that Γ' satisfies . . .," we may as well replace Γ by Γ' .

Let \mathfrak{g} be the Lie algebra of the group of real points $G(\mathbb{R})$ of G ; it is a semisimple Lie algebra, so the Cartan–Killing form B (induced from the canonical bilinear form $(X, Y) \mapsto \text{tr}(XY)$ on $\text{End}_{\mathbb{R}}(\mathfrak{g})$ via $ad: \mathfrak{g} \rightarrow \text{End}_{\mathbb{R}}(\mathfrak{g})$) is nondegenerate. The form B is characteristic (i.e. for all $\theta \in \text{Aut}_{\text{Lie alg}}(\mathfrak{g})$ we have $B(X, Y) = B(\theta X, \theta Y)$) and is invariant (i.e. $B([X, Y], Z) = B(X, [Y, Z])$).

Let H be an inner product on \mathfrak{g} , i.e. a positive definite real symmetric bilinear form. If we think of a bilinear form also as a map $\mathfrak{g} \rightarrow \mathfrak{g}^*$ (or as a matrix formed with respect to a basis of \mathfrak{g} and its dual basis of \mathfrak{g}^*), then an expression like $H^{-1}B$ makes sense as an \mathbb{R} -linear endomorphism of \mathfrak{g} (and is independent of the choice of basis). We say that H is *compatible* (with the Lie bracket) if $-H^{-1}B$ is a Lie algebra automorphism of \mathfrak{g} . One consequence of compatibility is that $\theta := -H^{-1}B$ is a Cartan involution of \mathfrak{g} , as we see now. The fact that $\theta^2 = 1$ follows from $\theta B \theta = B$ (B is characteristic) and $\theta B = -BH^{-1}B = B\theta$. Letting $\mathfrak{k} = (+1)$ -eigenspace of θ and $\mathfrak{p} = (-1)$ -eigenspace, we see from $B\theta = -H < 0$ that B is negative definite on \mathfrak{k} and positive definite on \mathfrak{p} . It follows from $\theta B \theta = B$ that \mathfrak{k} and \mathfrak{p} are orthogonal with respect to B , and thus that θ is a Cartan involution.

Another consequence of compatibility is that H is compatible with B in the sense of [G, Definition 7.3]. For the equation $\theta^2 = 1$ may be rewritten as $\theta B \theta = B$, which says that B is an isometry $\mathfrak{g} \rightarrow \mathfrak{g}^*$ with respect to the inner products H on \mathfrak{g} and H^{-1} on \mathfrak{g}^* .

Now let X be the space of all compatible inner products on \mathfrak{g} ; it is a closed subspace of the space of all inner products on \mathfrak{g} . Suppose F is any Lie algebra automorphism of \mathfrak{g} ; then $F B F = B$ because B is characteristic. For any $H \in X$ we

