

The K -Theory of Semilinear Endomorphisms

DANIEL R. GRAYSON*

Department of Mathematics, University of Illinois, Urbana, Illinois 61801

Communicated by Richard G. Swan

Received January 23, 1985

In this paper we study the K -theory of semilinear endomorphisms and automorphisms over noncommutative rings. For commutative rings and linear endomorphisms we did this in [G3].

In Section 4 we produce an exact sequence (4.6) involving the K -groups of semilinear automorphisms over a field. The main tool is the introduction of the "twisted projective line," together with the fact that it admits an interesting localization at $\{0, \infty\}$. In Section 5 we use the Frobenius on an algebraically closed field to produce an example of a semilocal domain B with nonzero radical J so that $K_i(B) \cong K_i(B/J)$, $i > 0$.

In Sections 1 and 2 we give another application of the twisted projective line: we prove the natural generalization (2.3) to the higher K -groups of the results of Farrell and Hsiang [FH] about Whitehead groups of twisted Laurent polynomial rings. The proof is a straightforward rewriting of Quillen's proof of the Fundamental Theorem [G2] (in which the adjoined variable was central). The difference between our proof and Ranicki's proof in [R, pp. 427–428] is that we emphasize the role of the twisted projective line, and we identify the group $F_i(\varphi)$ as the homotopy group of the homotopy fiber of the map $1 - \varphi^*$.

Other proofs are available. When the ground ring is regular noetherian, the theorem is an exercise in [Q1, pp. 114–122]. One could also obtain a proof by rewriting the proof of Theorem 18.1 of [W], which is much more general.

1. THE TWISTED PROJECTIVE LINE

A *right denominator set* S in a ring R is a subset with the following properties [St, p. 52]:

$$(S1) \quad 1 \in S,$$

* This work has been supported by the NSF under Grant MCS 82-02692. I thank the referees for their useful amendments.

- (S2) $s_1, s_2 \in S \Rightarrow s_1 s_2 \in S,$
- (S3) $s_1 \in S, a \in R \Rightarrow \exists b \in R, s_2 \in S: s_1 b = a s_2,$
- (S4) $s_1 \in S, a \in R, s_1 a = 0 \Rightarrow \exists s_2 \in S: a s_2 = 0.$

These conditions are the most general which ensure that the ring of right fractions RS^{-1} exists. If the elements of S are nonzero divisors (as will be the case here) then (S4) can be omitted. If $S = \{s^n: n \geq 0\}$ for some s , we write $R[s^{-1}] = RS^{-1}$.

The axioms for a *left denominator set* are analogous. If S is both a right and a left denominator set, we will call it a *denominator set*; in this case the ring of left fractions $S^{-1}R$ is isomorphic to RS^{-1} .

We let k be a (not necessarily commutative) ring, and φ an automorphism of k . The twisted polynomial ring $R^+ := k[T; \varphi]$ is the ring of polynomials $a_n T^n + \dots + a_0, a_i \in k$, where multiplication satisfies a $T = T\varphi(a)$. The multiplicative set generated by T is a denominator set, so the localization $R^\pm := k[T, T^{-1}; \varphi] := k[T; \varphi][T^{-1}]$ is defined; we see that $k[T^{-1}, (T^{-1})^{-1}; \varphi^{-1}] = k[T, T^{-1}; \varphi]$, so R^\pm is also a localization of $R^- := k[T^{-1}; \varphi^{-1}]$.

We define a *right X-module* M to be a triple $M = (M^+, M^-, \theta_M)$, where M^+ is a right R^+ -module, M^- is a right R^- -module, and $\theta_M = M^+[T^{-1}] \xrightarrow{\sim} M^-[(T^{-1})^{-1}]$ is an isomorphism of right R^\pm -modules. Here $X = \mathbb{P}^1(\varphi)$ denotes the "twisted projective line" with respect to k and φ and remains undefined. A *map* $f: M_1 \rightarrow M_2$ of X -modules is a pair $f^+: M_1^+ \rightarrow M_2^+, f^-: M_1^- \rightarrow M_2^-$ of homomorphisms with $\theta_{M_2} \cdot f^+ = f^- \cdot \theta_{M_1}$.

The category of right X -modules is an abelian category. Let \mathcal{M}_X denote the exact category of right X -modules M for which M^+ and M^- are finitely generated; it is an abelian category if R^+ and R^- are noetherian, and thus if k is noetherian (according to [FH, Lemma 24]). Let \mathcal{P}_R denote the exact category of finitely generated projective right R -modules, and let \mathcal{P}_X be the exact category of "vector bundles on X ," i.e., those X -modules M where $M^+ \in \mathcal{P}_{R^+}$ and $M^- \in \mathcal{P}_{R^-}$. Let

$$K_* X := K_* \mathcal{P}_X.$$

If R is R^+, R^- , or R^\pm , then φ extends to an automorphism of R by setting $\varphi(T) = T$. Tensor product gives an exact functor $\varphi^*: \mathcal{P}_R \rightarrow \mathcal{P}_R$. Define $N\langle n \rangle = (\varphi^{-n})^*(N)$ for $N \in \mathcal{P}_R$ and $n \in \mathbb{Z}$. One may also obtain $N\langle n \rangle$ from N by replacing the scalar multiplication with $x * f = x\varphi^n(f)$ for $x \in N$ and $f \in R$. If M is an X -module, we let $M\langle n \rangle = (M^+\langle n \rangle, M^-\langle n \rangle, \theta_M)$.

For k -modules V and W , a φ -semilinear map $f: V \rightarrow W$ is an additive map satisfying $f(va) = f(v)\varphi(a)$ for $v \in V, a \in k$. This is the same as a

k -linear map $V \rightarrow W\langle 1 \rangle$. If M is an R^+ -module, then right multiplication by T on M is a φ -semilinear endomorphism of the k -module underlying M , and all φ -semilinear endomorphisms of k -modules arise this way.

If M is an X -module, we define $M(n) := (M^+, M^-\langle -n \rangle, \theta_M \circ \rho(T^{-n}))$, where $\rho(T^{-n})$ denotes right multiplication by T^{-n} . One checks that $M(n) \in \mathcal{P}_X$, and $M(m)(n) = M(m+n)$.

If V is a k -module, define an X -module $V(0) := (V \otimes_k R^+, V \otimes_k R^-, 1)$, and X -modules $V(n) := V(0)(n)$. Let $h_n: \mathcal{P}_k \rightarrow \mathcal{P}_X$ denote the exact functor $h_n(V) = V(n)$.

THEOREM 1.1. *The map*

$$(h_{0*}, h_{1*}): K_i k \oplus K_i k \rightarrow K_i X$$

is an isomorphism. The relation $h_{m} + h_{m+2}\langle 1 \rangle_* = h_{m+1,*} + h_{m+1}\langle 1 \rangle_*$ holds for all $m \in \mathbb{Z}$.*

Proof. The proof can be done essentially as in [Q1, Theorem 3.1, Sect. 8, p. 143]; the only change is that T is no longer central. Multiplication by T on an R^+ -module N is no longer an R -linear endomorphism of N , but is an R -linear map $N \rightarrow N\langle 1 \rangle$. Thus, one rewrites Quillen's proof by inserting notations like " $\langle n \rangle$ " in appropriate spots to preserve linearity of the maps involved. For example, the canonical exact sequence

$$0 \rightarrow \mathcal{O}(m) \rightarrow \mathcal{O}(m+1)^2 \rightarrow \mathcal{O}(m+2) \rightarrow 0$$

becomes

$$0 \rightarrow V(m) \rightarrow V(m+1) \oplus V(m+1)\langle 1 \rangle \rightarrow V(m+2)\langle 1 \rangle \rightarrow 0$$

for any $V \in \mathcal{P}_k, m \in \mathbb{Z}$.

Q.E.D.

2. LOCALIZATION

In this section we discuss localization theorems for K -theory in the twisted projective line. This allows us to relate the K -groups of the projective line with those of $R^+, R^-,$ and R^\pm . In the commutative case, the result obtained is the "Fundamental Theorem" of Bass, generalized by Quillen to the higher K -groups. In the case at hand, we obtain the result of Farrell and Hsiang and generalize it to apply to the higher K -groups.

Let \mathcal{H}^+ denote the exact category of X -modules M which admit a resolution of length 1 by vector bundles of X and for which $M^- = 0$. This category is equivalent to the category of finitely generated R^+ -modules N

of projective dimension 1 such that $N[T^{-1}] = 0$, for a resolution of N may be begun with a free R^+ -module (which extends to X). Observe that for any right R^+ -module P , the subgroup $P \cdot T^i$ is an R^+ -submodule; moreover, if $P = R^+$, then $P/P \cdot T^i$ is a free k -module on the generators $1, T, \dots, T^{i-1}$. Now the argument of [G2, p. 236] shows that any N in \mathcal{H}^+ is projective as k -module, so \mathcal{H}^+ is equivalent to the category $\underline{\text{Nil}}(\varphi)$ whose objects are pairs (V, f) with $V \in \mathcal{P}_k$ and $f: V \rightarrow V$ a nilpotent φ -semilinear endomorphism, $f(va) = f(v)\varphi(a)$. (In the untwisted case $\varphi = 1$ this equivalence is implicit in [B, proof of the fundamental theorem] and explicit in [Si].)

The exact functors

$$\begin{aligned} \mathcal{P}_k &\rightarrow \underline{\text{Nil}}(\varphi), & \underline{\text{Nil}}(\varphi) &\rightarrow \mathcal{P}_k \\ V &\mapsto (V, 0), & (V, f) &\mapsto V \end{aligned}$$

allow one to split

$$K_i \underline{\text{Nil}}(\varphi) = K_i k \oplus \underline{\text{Nil}}_i(\varphi),$$

defining $\underline{\text{Nil}}_i(\varphi)$.

The ring homomorphisms

$$\begin{aligned} k &\rightarrow R^+, & R^+ &\rightarrow k \\ a &\rightarrow a, & f(T) &\rightarrow f(0) \end{aligned}$$

allow one to split

$$K_i R^+ = K_i k \oplus NK_i(\varphi),$$

defining $NK_i(\varphi)$. Similarly,

$$K_i R^- = K_i k \oplus NK_i(\varphi^{-1}).$$

THEOREM 2.1. *There are localization exact sequences*

- (a) $\dots K_{i+1} R^\pm \rightarrow K_i \mathcal{H}^\pm \rightarrow K_i R^\pm \rightarrow K_i R^\pm \dots$,
- (b) $\dots K_{i+1} R^- \rightarrow K_i \mathcal{H}^+ \rightarrow K_i X \rightarrow K_i R^- \dots$, and
- (c) $NK_i(\varphi^{-1}) = \underline{\text{Nil}}_{i-1}(\varphi)$.

Proof. Part (a) was proved in [G1]. For part (b) one checks that the proof in [G2, Theorem on p. 222] can be carried over into this context,

using the preliminary material about the twisted projective line presented above. One interprets the notation from [G2] as follows:

$$\begin{aligned}
 j^*M &:= M^- \\
 j_*M^- &:= (M^- \otimes R^\pm, M^-, 1) \\
 I^{-n}M &:= (M^+ \cdot T^{-n}, M^-, 1) \\
 &\subseteq j_*j^*M.
 \end{aligned}$$

Part (c) follows from (b) as in [G2].

Q.E.D.

Remark 2.2. If k is commutative, or if we are given an isomorphism $k \cong k^{\text{op}}$, then there is an isomorphism $(R^+)^{\text{op}} \cong R^-$. It follows from [Q1, (13) on p. 104] that $K_iR^+ = K_iR^-$, and thus $NK_i(\varphi) \cong NK_i(\varphi^{-1})$ and $\text{Nil}_i(\varphi) \cong \text{Nil}_i(\varphi^{-1})$. There is also an equivalence $\underline{\text{Nil}}(\varphi)^{\text{op}} \cong \underline{\text{Nil}}(\varphi^{-1})$ defined by $(V, f) \mapsto (V^*, f')$, where $V^* = \text{Hom}_k(V, k)$ and $f' = \varphi_*^{-1} \circ f^*$. The isomorphism $\text{Nil}_i(\varphi) \cong \text{Nil}_i(\varphi_*^{-1})$ that this equivalence provides is probably the same as the other one.

Remark. One can use Quillen's dévissage and resolution theorems to prove that $\text{Nil}_*(\varphi) = 0$ when k is regular noetherian, thereby recovering his result that $K_iR^+ \cong K_ik$.

Define $F_i(\varphi) = \pi_i \Omega(K(k) \rightarrow^{1-\varphi^*} K(k))$, where $K(k)$ is the space $\Omega BQ\mathcal{P}_k$, whose homotopy groups are the K -groups, and where $\Omega(X \rightarrow Y)$ denotes the homotopy fiber of a map. If $\varphi = 1$, then $F_i(\varphi) = K_i(k) \oplus K_{i+1}(k)$. Notice, also, that $F_i(\varphi^{-1}) = F_i(\varphi)$.

THEOREM 2.3. *There is, for $i \geq 1$, a canonical isomorphism*

$$K_iR^\pm \cong F_{i-1}(\varphi) \oplus \text{Nil}_{i-1}(\varphi) \oplus \text{Nil}_{i-1}(\varphi^{-1}).$$

Remark. For $i = 1$, this theorem was proved by Farrell and Hsiang and by Siebenmann.

Proof. There is a restriction map from the sequence 2.1(b) to 2.1(a), which is the identity on $K_i\mathcal{H}^+$. A diagram chase yields a Mayer-Vietoris-type exact sequence

$$\dots K_{i+1}R^\pm \rightarrow K_iX \rightarrow K_iR^+ \oplus K_iR^- \rightarrow K_iR^\pm \dots$$

We rewrite the terms using (1.1) and 2.1(c) yielding

$$\begin{array}{c}
 \vdots \\
 K_{i+1}R^\pm \\
 \downarrow B \\
 K_i k \oplus K_i k \\
 \downarrow A \\
 K_i k \oplus \text{Nil}_{i-1}(\varphi^{-1}) \oplus K_i k \oplus \text{Nil}_{i-1}(\varphi) \\
 \downarrow \\
 K_i R^\pm \\
 \vdots
 \end{array}$$

The matrix of the map A is seen to be

$$\begin{pmatrix}
 1 & 1 \\
 0 & 0 \\
 1 & \varphi^* \\
 0 & 0
 \end{pmatrix}$$

and because $AB=0$, we see that the matrix of B is

$$\begin{pmatrix}
 g \\
 -g
 \end{pmatrix}$$

for some map g . This allows us to split off a $K_i k$ factor, yielding

$$\dots K_{i+1}R^\pm \xrightarrow{g} K_i k \xrightarrow{\begin{pmatrix} 1-\varphi^* \\ 0 \\ 0 \end{pmatrix}} K_i k \oplus \text{Nil}_{i-1}(\varphi^{-1}) \oplus \text{Nil}_{i-1}(\varphi) \rightarrow K_i R^\pm \dots$$

Consider the diagram

$$\begin{array}{ccccc}
 & & K_{i+1}(R^-) & \rightarrow & K_i(\mathcal{H}^+) \\
 & & \downarrow & & \parallel \\
 & & K_{i+1}(R^+) & \rightarrow & K_{i+1}(R^\pm) \rightarrow K_i(\mathcal{H}^+) \\
 & \downarrow & \downarrow & & \downarrow \\
 & & K_i(\mathcal{H}^-) & = & K_i(\mathcal{H}^-)
 \end{array}$$

with exact rows and columns. Application of the decompositions we know so far gives

$$\begin{array}{ccccc}
 & & K_{i+1}k \oplus \text{Nil}_i(\varphi) & \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} & K_i k \oplus \text{Nil}_i(\varphi) \\
 & & \downarrow & & \parallel \\
 K_{i+1}k \oplus \text{Nil}_i(\varphi^{-1}) & \rightarrow & K_{i+1}R^\pm & \longrightarrow & K_i k \oplus \text{Nil}_i(\varphi) \\
 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \downarrow & & \downarrow & & \\
 K_i k \oplus \text{Nil}_i(\varphi^{-1}) & = & K_i k \oplus \text{Nil}_i(\varphi^{-1}) & &
 \end{array}$$

It follows that

$$K_{i+1}R^\pm \cong ? \oplus \text{Nil}_i(\varphi) \oplus \text{Nil}_i(\varphi^{-1})$$

and we get an exact sequence

$$\dots \rightarrow ? \rightarrow K_i k \xrightarrow{1-\varphi^*} K_i k \rightarrow \dots$$

In order to identify “?” with $F_i(\varphi)$, we argue with the underlying spaces. We get a map of fibrations

$$\begin{array}{ccccc}
 \Omega K(R^\pm) & \rightarrow & K(k) & \xrightarrow{\begin{pmatrix} 1-\varphi^* \\ \text{pt.} \\ \text{pt.} \end{pmatrix}} & K(k) \times NK(\varphi^{-1}) \times NK(\varphi) \\
 \begin{matrix} \uparrow s \\ \downarrow t \end{matrix} & & \downarrow 1 & & \downarrow pr_1 \\
 F(\varphi) & \rightarrow & K(k) & \xrightarrow{(1-\varphi^*)} & K(k)
 \end{array}$$

where the notations NK and F for spaces ought to be self-explanatory. The existence of the section s follows from Lemma 2.4 below. The spaces here are homotopy-everything H -spaces with additive inverses, so we may split

$$\Omega K(R^\pm) \cong F(\varphi) \times \Omega(t).$$

Moreover, the homotopy fibers of the three vertical maps above form a fibration which tells us that

$$\Omega(t) \cong \Omega NK(\varphi^{-1}) \times \Omega NK(\varphi).$$

Thus

$$\Omega K(R^\pm) \cong F(\varphi) \times \Omega NK(\varphi^{-1}) \times \Omega NK(\varphi).$$

Taking homotopy groups yields the result.

Q.E.D.

LEMMA 2.4. *Given maps of pointed spaces $f: A \rightarrow X$ and $g: A \rightarrow Y$, let*

$G = \Omega(A \rightarrow^g Y)$ and $F = \Omega(A \xrightarrow{(f, g)} X \times Y)$. A null homotopy $f \sim pt$ provides a section for the projection $F \rightarrow G$.

Proof. This follows immediately from the definition of the homotopy fiber, namely $G = A \times_Y Y^I \times_Y y_0$, where $\{y_0\}$ is the base point of Y .

Q.E.D.

3. DEFINING MODULES LOCALLY

In this section, we prove a version of the theorem from commutative algebra that says quasicohherent sheaves may be defined locally.

Suppose S and T are right denominator sets in R , and let $U := \langle S, T \rangle$ denote the multiplicative set they generate. It is easy to see that U is also a right denominator set.

It follows from the universal property for localizations that RU^{-1} is the pushout (in the category of rings) of the diagram $RS^{-1} \leftarrow R \rightarrow RT^{-1}$.

We call S and T compatible if $ST = TS (= U)$, or equivalently, the following axioms are satisfied:

$$(ST1) \quad s_1 \in S, t_1 \in T \Rightarrow \exists t_2 \in T, s_1 \in S \ s_1 t_2 = t_2 s_2,$$

$$(ST2) \quad t_1 \in T, s_1 \in S \Rightarrow \exists s_2 \in S, t_2 \in T \ t_1 s_1 = s_2 t_2.$$

LEMMA 3.1. *If S and T are compatible, then $(RS^{-1})T^{-1} \cong R(ST)^{-1} \cong (RT^{-1})S^{-1}$ are all isomorphic as rings.*

Proof. One checks that the image of T in RS^{-1} is a right denominator set, then the statement follows from the universal property of localization.

Q.E.D.

We introduce the following covering axiom for S and T .

$$(ST3) \quad s \in S \text{ and } t \in T \Rightarrow sR + tR = R.$$

This axiom implies that $RS^{-1} \times RT^{-1}$ is faithfully flat as left R -module. For if $0 = M \otimes_R (RS^{-1} \times RT^{-1}) = MS^{-1} \oplus MT^{-1}$ and $m \in M$, then $ms = mt = 0$ for some $s \in S, t \in T$, thus $m = 0$, and $M = 0$. Then one proves the following in the usual way.

PROPOSITION 3.2. *Suppose $S, T \subseteq R$ are right denominator sets which are compatible and satisfy the covering axiom. Then the category of (right) R -modules M is equivalent to the category of triples (P, Q, θ) , where P is an RS^{-1} -module, Q is an RT^{-1} -module, and $\theta: PT^{-1} \rightarrow QS^{-1}$ is an $R(ST)^{-1}$ -isomorphism.*

COROLLARY 3.3. *In the equivalence of Proposition 3.2, M is finitely generated (resp. finitely presented) iff P and Q are. If S and T are also left denominator sets, then M is finitely generated projective iff P and Q are.*

Proof. The proof of the first assertion is standard. For the second we consider the sequence

$$0 \rightarrow M \rightarrow MS^{-1} \oplus MT^{-1} \rightarrow M(ST)^{-1} \rightarrow 0,$$

which is exact because it becomes exact under localization by S or by T . The hypothesis implies that RS^{-1} , RT^{-1} , and $R((ST)^{-1})$ are all right and left flat over R , so M is also. Since M is flat and finitely presented, it follows from Lazard's theorem [La, Corollary 1.4] that M is projective.

Q.E.D.

4. A LOCALIZATION OF THE PROJECTIVE LINE

We now make the blanket assumption that k is a (skew) field. Let $S^+ \subseteq R^+$ be the multiplicative set of all nonzero polynomials, and let $S_0^+ \subseteq R^+$ be the multiplicative set of all polynomials with nonzero constant term.

LEMMA 4.1. *S^+ and S_0^+ are denominator sets (consisting only of nonzero divisors).*

Proof. First prove it for S^+ . Let R_j denote the polynomials of degree $\leq j$. Given $f \in R^+$ and $s \in S^+$, let $m = \deg f$, $n = \deg s$, and consider the map $R_m \oplus R_n \rightarrow R_{m+n}$ defined by $(u, v) \rightarrow fu - sv$. This k -linear map has nonzero kernel for dimension reasons. When $fu - sv = 0$, then $u \in S^+$ unless $u = v = 0$, for R^+ is an integral domain.

Next prove it for S_0^+ . Proceed as before: if $u(0) = 0$, then $v(0) = 0$ (because $s(0) \neq 0$), so we may divide u and v by a suitable power of T to achieve $u \in S_0^+$.

We've given the proof on the right side: the left side goes the same way. Q.E.D.

In the ring $B^+ := (S_0^+)^{-1}R^+ = R^+(S_0^+)^{-1}$, the multiplicative set generated by T still is a denominator set, so letting $B^\pm := B^+[T^{-1}]$, we see that $B^\pm = R^+(S^+)^{-1}$ is a skew field. Using by now obvious notation, we also have the ring $B^- := R^-(S_0^-)^{-1}$, and $B^\pm = B^-[(T^{-1})^{-1}]$. Define $B := B^+ \cap B^- \subseteq B^\pm$.

LEMMA 4.2. *B consists of all fractions fg^{-1} , with f and $g \in R^+$, $g(0) \neq 0$, and $\deg g \geq \deg f$.*

Proof. Write a typical element of $B \subseteq B^+$ in the form fg^{-1} with $f, g \in R^+, g \in S_0^+$. Let $n = \max(\deg f, \deg g)$, and let $G(T^{-1}) := g(T) T^{-n}$, $F(T^{-1}) := f(T) T^{-n}$ so that $fg^{-1} = FG^{-1}$, and $G, F \in R^-$. Since $FG^{-1} \in B^-$, we may write $FG^{-1} = JH^{-1}$ with $H, J \in R^-$ and $H \in S_0^-$. By definition of fractions, we may find K, L nonzero in R^- so that $GK = HL$ and $FK = JL$.

We may assume T^{-1} does not divide both K and L . Then if T^{-1} divides G , it follows that $T^{-1} \mid L$ and $T^{-1} \nmid K$, and so $T^{-1} \mid F$. But T^{-1} does not divide both F and G , so $T^{-1} \nmid G$, and $n = \deg g \geq \deg f$. Q.E.D.

Let $p_0: B^+ \rightarrow k$ be the ring homomorphism with $p_0(T) = 0$, and let $p_\infty: B^- \rightarrow k$ be the homomorphism with $p_\infty(T^{-1}) = 0$. If $p_0(fg^{-1}) \neq 0$, then $f(0) \neq 0$, so $gf^{-1} \in B^+$ and fg^{-1} is a unit in B^+ . Thus $I_0 = \ker p_0$ is a maximal (left, right, or 2-sided) ideal whose complement consists of units, and is the only maximal (left or right) ideal. The same remarks apply to $I_\infty = \ker p_\infty \subseteq B^-$. Thus the rings B^+, B^- are local.

Let $J_0 := I_0 \cap B, J_\infty := I_\infty \cap B$. If $fg^{-1} \in B$, and $p_0(fg^{-1}) \neq 0, p_\infty(fg^{-1}) \neq 0$, then it follows that $f(0) \neq 0$ and $\deg g = \deg f$, so fg^{-1} is a unit in B (by Lemma 4.2). It follows that J_0, J_∞ are the only maximal (left or right) ideals of B . For if C is another maximal left ideal, take $\beta \in C \setminus J_0$ and $\gamma \in C \setminus J_\infty$; one of $\beta, \gamma, \beta + \gamma$ is in $C \setminus (J_0 \cup J_\infty) = C \cap B^\times$, a contradiction. We conclude that the radical $J := \text{rad}(B) = J_0 \cap J_\infty$ and is the kernel of the surjective homomorphism

$$p = (p_0, p_\infty): B \rightarrow k \times k.$$

LEMMA 4.3. $B^+, B^-,$ and B^\pm are all left (or right) rings of fractions of B .

Proof. Suppose $fg^{-1} \in B^+$, with $g, f \in R^+$ and $g(0) \neq 0$. Let $b := \max\{0, \deg f - \deg g\}$, and $h = (1 + T)^b \cdot g$. Then $fg^{-1} = (fh^{-1})((1 + T)^{-b})^{-1}$, and $fh^{-1} \in B, (1 + T)^{-b} \in B$, which shows B^+ is a right ring of fractions of B . The proof for B^- is similar (replace T by T^{-1}), as is the proof on the left side. Since we didn't use the condition $g(0) \neq 0$ in arranging $\deg h \geq \deg f$, the proofs for B^+ and B^- combine to show B^\pm is a localization of B . Q.E.D.

According to the lemma, we may write

$$T^+ := R \cap (R^+)^\times = \{fg^{-1} \mid g(0) \neq 0, f(0) \neq 0, \deg f \leq \deg g\}$$

$$T^- := B \cap (B^-)^\times = \{fg^{-1} \mid g(0) \neq 0, \deg f - \deg g\}$$

$$T^\pm := B \cap (B^\pm)^\times = B \setminus \{0\}$$

$$B^+ = (T^+)^{-1} B$$

$$B^- = (T^-)^{-1} B$$

$$B^\pm = (T^\pm)^{-1} B.$$

LEMMA 4.4. *Proposition 3.2 and all of Corollary 3.3 apply to the multiplicative sets T^+ and T^- in the ring B .*

Proof. Given $fg^{-1} \in T^\pm$ with $f, g \in R^+$, we may write $fg^{-1} = (1+T)^{-a} (((1+T)^a f) g^{-1})$ with $a = \deg g - \deg f \geq 0$. This makes $(1+T)^a fg^{-1} \in T^-$ and $(1+T)^{-a} \in T^+$, so $T^\pm = T^+ T^-$. By symmetry (writing denominators on the other side) we see that $T^\pm = T^- \cdot T^+$, and thus T^+ and T^- are compatible.

The covering condition follows from $T^+ = B \setminus J_0$, $T^- = B \setminus J_\infty$, and the fact that J_0 and J_∞ are the only maximal right ideals of B . Q.E.D.

Remark. It follows that B is a ring of global dimension 1, because B^+ and B^- are

COROLLARY 4.5. *There are exact functors $\mathcal{M}_X \rightarrow \mathcal{M}_B$ and $\mathcal{P}_X \rightarrow \mathcal{P}_B$ defined by $(M^+, M^-, \theta) \rightarrow$ pullback of $(M^+ \otimes_{R^+} B^+, M^- \otimes_{R^-} B^-, \theta \otimes 1)$.*

We may think of this functor as a localization functor. Indeed, as in the commutative case, we may think of B as the semilocal ring at $\{0, \infty\}$ in the projective line.

We denote the functors of (4.5) with $M \rightarrow M \otimes_X B$, for $M \in \mathcal{M}_X$. Let \mathcal{H} denote the exact category of all those X -modules M which have a resolution of length 1 on X by vector bundles on X , and for which $M \otimes_X B = 0$.

Define $\text{Aut}(\varphi)$ to be the exact category consisting of all pairs (V, f) with $V \in \mathcal{P}_k$ and $f: V \rightarrow V$ a φ -semilinear automorphism, $f(va) = f(v)\varphi(a)$. An arrow $(V, f) \rightarrow (V', f')$ is a map $g: V \rightarrow V'$ with $gf = f'g$, as usual.

THEOREM 4.6. (a) *There is a long exact "localization" sequence*

$$\dots K_i \mathcal{H} \rightarrow K_i X \rightarrow K_i B \rightarrow K_{i-1} \mathcal{H} \dots$$

(b) *There is an equivalence $\mathcal{H} \cong \text{Aut}(\varphi)$ of exact categories.*

Proof. (a) We reread Quillen's proof of the localization theorem for projective modules [G2, p. 229] to verify that it works in our context. The crucial Lemma 2 there is rephrased as follows: for each $N \in \mathcal{P}_B$, the category C_N of pairs (M, β) , with $M \in \mathcal{P}_X$, and β an isomorphism $\beta: M \otimes_X B \xrightarrow{\sim} N$, is equivalent to a filtering ordered set. (One may compare C_N with \mathcal{L}_W of [G1].) To convince ourselves of this statement, we first, for

each M , replace M^+ by its isomorphic image in N^+ , and similarly for M^- . This gives a retraction of C_N onto a partially ordered set \mathcal{D}_N , consisting of certain "submodules" of N . Since $M_1 + M_2 \in \mathcal{D}_N$ when $M_1, M_2 \in \mathcal{D}_N$, we see that \mathcal{D}_N is filtering. (The reason $M_1 + M_2 \in \mathcal{D}_N$ is that R^+ and R^- are (noncommutative) Euclidean domains, for which any finitely generated torsion free module is projective.)

(b) A functor $F: \mathcal{H} \rightarrow \text{Aut}(\varphi)$ can be defined by $M \rightarrow (M^+, \text{multiplication by } T)$. A functor $G: \text{Aut}(\varphi) \rightarrow \mathcal{H}$ can be defined by $(V, f) \rightarrow (V_f, V_f, 1)$, where V_f denotes the R^+ -module whose underlying k -module is V , but on which T acts as f , and where V_f also denotes the R^- -module which is V with T^{-1} acting as f^{-1} . Certain details must be checked, the only obvious one being that $F \circ G = 1$.

To see that $G \circ F = 1$, we must verify that for $M \in \mathcal{H}$, T acts invertibly on M^+ and T^{-1} acts invertibly on M^- , so that $M^+ \cong M^+[T^{-1}] \cong M^-[(T^{-1})^{-1}] \cong M^-$. From $M^+(S_0^+)^{-1} = 0$ it follows that for any $x \in M^+$ there exists $s \in S_0^+$ with $xs = 0$. Writing $s = a_0 + a_1T + \dots + a_nT^n$ ($a_0 \neq 0$) we see that $x = (-x(a_1 + \dots + a_nT^{n-1}) \varphi^{-1}(a_0^{-1}))T$, showing that multiplication by T is surjective. For injectivity, the assumption $xT = 0$ implies $xa_0 = 0$, whence $x = 0$.

To see that F is well-defined, we must check that if $M \in \mathcal{H}$, then M^+ is a finite-dimensional k -vector space; this is clear, for we may express M^+ as a quotient of $(R/sR)^j$, some $s \in S_0^+$, some j .

To see that G is well-defined we must, given $(V, f) \in \text{Aut}(\varphi)$ and $v \in V_f$, locate $s \in S_0^+$ so that $v \cdot s = 0$. This is done in the usual way, by considering $\{v, vT, vT^2, \dots\} \subseteq V$. We must also check that $G(V, f)$ has a resolution of length one by X -vector bundles; it is easy to establish the exactness of the sequence

$$0 \rightarrow V(-1) \xrightarrow{g} V(0) \xrightarrow{k} G(V, f) \rightarrow 0,$$

where k is the obvious map, and g consists of

$$\begin{aligned} V \otimes R^+ &\rightarrow V \otimes R^+ \\ v \otimes p &\rightarrow f^{-1}(v) \otimes Tp \quad v \otimes p \end{aligned}$$

and

$$\begin{aligned} V \otimes R^- \langle 1 \rangle &\rightarrow V \otimes R^- \\ v \otimes q &\rightarrow f^{-1}(v) \otimes \varphi^{-1}(q) - v \otimes qT^{-1}. \end{aligned}$$

This is the characteristic sequence of the semilinear automorphism f (cf. B, p. 630; G3, p. 442; and FH, Lemma 9). Q.E.D.

Remark. If $\varphi = 1$, then as in [G3], $K_i \text{Aut}(\varphi)$ contains $K_i k$ as a direct

factor, because $(V, 1_V) \in \text{Aut}(\varphi)$ for any $V \in \mathcal{P}_k$. If $\varphi \neq 1$ then this no longer works. For this reason, it is not possible to describe $K_i \text{Aut}(\varphi)$ as in [G3, Theorem 2]. The localization sequence for $R^+ \rightarrow B^+$ does, however, split into short exact sequences, yielding the decomposition

$$K_i B^+ = K_i R^+ \oplus K_{i-1} \text{Aut}(\varphi).$$

5. THE CASE $\varphi = \text{FROBENIUS}$

In this final section we assume k is an algebraically closed (commutative) field of characteristic p , and φ is the Frobenius, $\varphi(a) = a^p$. We let \mathbb{F}_p denote the prime field. The functor $L: \mathcal{P}_{\mathbb{F}_p} \rightarrow \text{Aut}(\varphi)$ defined by $W \rightarrow (W \otimes k, 1_W \otimes \varphi)$ is known to be an equivalence of categories [Q1, p. 115] or [L]. This “deep descent” presents the possibility of using (4.6) and (1.1) to compute $K_i B$.

Remark. To extract the statement that L is an equivalence one proceeds as follows. Given $(V, f) \in \text{Aut}(\varphi)$, choose a basis of V and let A be the matrix of f with respect to that basis; then [L] provides a matrix B with $B^{(p)} B^{-1} = A$. One can check that B provides a change of basis for V so that f fixes each element of the basis. This shows that the functor $\text{Aut}(\varphi) \rightarrow \mathcal{P}_{\mathbb{F}_p}$ defined by $(V, f) \rightarrow \{v \in V \mid f(v) = v\}$ is well-defined and an inverse equivalence for L .

THEOREM 5.1. *Under the assumptions made above, the map $K_i(B) \rightarrow K_i(B/J)$ is an isomorphism for $i > 0$, and thus $K_i B \simeq K_i k \times K_i k$.*

Proof. We make explicit the dotted arrow in the following diagram:

$$\begin{array}{ccc} K_i \mathcal{A} & \xrightarrow{i^*} & K_i X \\ L^* \uparrow \parallel & & \uparrow \parallel (k_0^*, k_1^*) \\ K_i \mathbb{F}_p & \cdots \cdots \cdots & K_i k \times K_i k \end{array}$$

The characteristic sequence of (4.6) is natural in V , so we find that $i^* \circ L^* = (h_0^* - h_{-1}^*) \circ j^*$, where we let j denote the inclusion $\mathbb{F}_p \rightarrow k$. The natural exact sequence of (1.1) yields $h_0^* - h_{-1}^* = (\varphi^{-1})^* (h_1^* - h_0^*)$. The isomorphism $V\langle 1 \rangle \otimes R^+ \rightarrow V \otimes R^+\langle 1 \rangle$ defined by $v \otimes p \rightarrow v \otimes \varphi(p)$, with a similar one for R^- , provides an isomorphism $V(n)\langle m \rangle = V\langle m \rangle(n)$; thus $\varphi^* h_n^* = h_n^* \varphi^*$. Since $\varphi^* j^* = j^*$ we get $i^* \circ L^* = (h_1^* - h_0^*) \circ j^*$, so the dotted arrow is (\bar{j}^*) . The matrix of the composite map

$$K_i k \times K_i k \rightarrow K_i X \rightarrow K_i B \rightarrow K_i B/J = K_i k \times K_i k$$

