These problems are due on Friday, April 10th in class, and will be graded on clarity of exposition as well as correctness. Some of them have been poached from Brown’s book, *Topology and Groupoids*. I encourage you to work in small groups, but what you hand in must be written in your own words.

1. **Universal property of the subspace topology.** Let $X$, $Y$ be topological spaces, and $A \subset X$ a subspace of $X$. Denote the inclusion map by $i: A \to X$. Show that $i$ is continuous, and $f: Y \to A$ is continuous if and only if $i \circ f$ is continuous.

2. **Gluing continuous functions.** Let $X$ be a topological space. An open cover of $X$ is a set $U$ of open subsets of $X$ such that $\bigcup_{U \in U} U = X$. A closed cover of $X$ is a set $F$ of closed subsets of $X$ such that $\bigcup_{F \in F} F = X$.

   (a) Let $X$ be a topological space, and $U$ an open cover of $X$. Show that a subset $A$ of $X$ is open in $X$ if and only if the intersection $A \cap U$ is open in $U$ (for the subspace topology on $U$) for each $U \in U$.

   (b) Let $Y$ be another topological space. Show that a function $f: X \to Y$ is continuous if and only if the restriction $f|_U: U \to Y$ is continuous for each $U \in U$ (where we take the subspace topology on $U$).

   (c) Let $F$ be a finite closed cover of $X$ (i.e., $F$ has finitely many elements). Show that a function $f: X \to Y$ is continuous if and only if the restriction $f|_F: F \to Y$ is continuous for each $F \in F$ (where we take the subspace topology on $F$).

   (d) Let $X, Y$ be topological spaces, and $H: X \times [0, 2] \to Y$ be a function such that $H|_{X \times [0, 1]}: X \times [0, 1] \to Y$ and $H|_{X \times [1, 2]}: X \times [1, 2] \to Y$ are both continuous. Show that $H$ is continuous.

3. **$T_1$ and Hausdorff spaces.** A topological space $X$ is called $T_1$ if, for each $x \in X$, the set $\{x\}$ is closed. We call $X$ Hausdorff if distinct points of $X$ have disjoint neighborhoods. Prove that a metric space is Hausdorff, and that a Hausdorff space is $T_1$.

4. **Quotient topology.** Let $X$ be a topological space, $Y$ be a set, and $q: X \to Y$ a surjective function. A subset $U \subseteq Y$ is open in the quotient topology on $Y$ if and only if $q^{-1}(U)$ is open in $X$. Show that this defines a topology on $Y$, and the function $q$ is continuous with respect to this topology.

5. **Line with two origins.** Take the discrete topology on $\{0, 1\}$, the standard topology on $\mathbb{R}$, and consider the equivalence relation $\sim$ on $\mathbb{R} \times \{0, 1\}$ generated by setting $(x, 0) \sim (x, 1)$ for $x \in \mathbb{R} \setminus \{0\}$. Let $X$ be the quotient $X := (\mathbb{R} \times \{0, 1\})/\sim$ equipped with the quotient topology for the map $q: (\mathbb{R} \times \{0, 1\}) \to (\mathbb{R} \times \{0, 1\})/\sim$. The resulting space is commonly called the line with two origins. Show that every singleton subset of $X$ is closed. Show that $X$ is not Hausdorff.

6. **Connected spaces.** Let $X$ be a topological space. A separation of $X$ is a pair $U, V$ of non-empty disjoint open subsets of $X$ such that $A \cup B = X$. We say $X$ is connected if there does not exist a separation of $X$. 

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(a) Consider the discrete topology on \{0, 1\}. Show that a space \(X\) is connected if and only if any continuous function from \(X\) to \{0, 1\} is constant.

(b) Assume \(f: X \rightarrow Y\) is a continuous map, and \(X\) is connected. Show that \(f(X)\) is connected (in the subspace topology).

(c) Show that any interval \([a, b]\) in \(\mathbb{R}\) is connected. [Hint: Assuming that \(U, V\) is a separation of \([a, b]\) such that \(a \in U\) and \(b \in V\) consider the element \(\sup(U)\).]

(d) Prove the intermediate value theorem: if \(X\) is a connected space, \(a, b \in X\) and \(f: X \rightarrow \mathbb{R}\) is a continuous map, then \([f(a), f(b)]\) is contained in the image of \(f\).

7. **The topologist’s sine curve.** Let \(X \subset \mathbb{R}^2\) be the graph of the function \(f(x) = \sin(\pi/x)\) for \(x \neq 0\). Show \(X\) is not closed as a subspace of \(\mathbb{R}^2\). What is its closure?

8. **Topologies on sets with few elements.** List all topologies on a set with 0, 1, 2, and 3 elements.

9. **Neighborhood basis.** A basis for the neighborhood at \(x \in X\) is a set \(B(x)\) of neighborhoods of \(x\) such that if \(N\) is a neighborhood of \(x\) then \(N\) contains some \(B \in B(x)\) as a subset. If we have such a basis \(B(x)\) for each \(x \in X\), then the function \(B\) defined by \(x \mapsto B(x)\) is called a basis for the neighborhoods of \(X\). A topological space \(X\) is said to satisfy the first axiom of countability if there is a basis \(B\) for the neighborhoods of \(X\) such that \(B(x)\) is countable for each \(x \in X\). Prove that the following satisfy the first axiom of countability: \(\mathbb{R}\) with the standard topology, \(\mathbb{Q}\) with the subspace topology for \(\mathbb{Q} \subset \mathbb{R}\) (and \(\mathbb{R}\) with the standard topology), a discrete space, a space with a countable number of open sets.

10. **A non-embedding.** Let \(S^1 := \{z \in \mathbb{C} : |z| = 1\}\) and \(\alpha \in \mathbb{R}\) an irrational number. Prove that the map \(\mathbb{R} \rightarrow S^1 \times S^1, t \mapsto (e^{2\pi i t}, e^{2\pi i t})\) is not an embedding.

11. **The \(p\)-adic topology.** Let \(X = \mathbb{Z}\) and let \(p\) be a fixed integer. A set \(N \subset \mathbb{Z}\) is a \(p\)-adic neighborhood of \(n \in \mathbb{Z}\) if \(N\) contains the integers \(n + mp\) for some \(r\) and all \(m = 0, \pm1, \pm2, \ldots\) (so that in a given neighborhood \(r\) is fixed but \(m\) varies). Prove that the \(p\)-adic neighborhoods form a neighborhood topology on \(\mathbb{Z}\), the \(p\)-adic topology.

12. **The \(p\)-adic metric.** Let \(p\) be a prime number. For each \(n \in \mathbb{N}\) define \(v_p(n)\) to be the exponent of \(p\) in the decomposition of \(n\) into prime numbers. If \(x = \pm m/n\) is any non-zero rational number, with \(m, n \in \mathbb{N}\), define \(v_p(x) := v_p(m) - v_p(n)\). Finally, if \(x, y\) are rational numbers define

\[
    d(x, y) = \begin{cases} 
      p^{-v_p(x-y)} & x \neq y \\
      0 & x = y.
    \end{cases}
\]

(a) Prove that \(d\) is a metric on \(\mathbb{Q}\) and that \(d\) satisfies the following strong form of the triangle inequality \(d(x, z) \leq \max\{d(x, y), d(y, z)\}\). The induced topology on \(\mathbb{Q}\) is called the \(p\)-adic topology.

(b) Prove that the topology induced by \(d\) on \(\mathbb{Z}\) is the \(p\)-adic topology of the previous problem.

(c) Justify the following statement: in the \(p\)-adic topology on \(\mathbb{Q}\), small rational numbers are those which are multiples of large powers of \(p\).