On drop property for convex sets

by

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Let $(X, \|\cdot\|)$ be a real Banach space. Let C be a closed convex set in X . By a drop $D(x, C)$ determined by a point $x \in X$, $x \notin C$, we shall mean the convex hull of the set $\{x\} \cup C$. We say that C has the drop property if $C \neq X$ and if for every nonvoid closed set A disjoint with C , there exists a point $a \in A$ such that $D(a, C) \cap A = \{a\}$. For a given C a sequence $\{x_n\}$ in X will be called a stream if $x_{n+1} \in D(x_n, C) \setminus C$ (cf. [6]).

When the set A has a positive distance from C , a variety of “Drop theorems” has been obtained in [1, 2, 3] and [8]. If C is the closed unit ball and has the drop property then we say that the norm $\|\cdot\|$ has the drop property [9]. Norms with the drop property have been investigated in papers [4, 6] and [9]. The drop property for closed bounded sets has been considered in [5]. There was proved that a bounded closed convex symmetric set having the drop property is compact or has a non-empty interior. We shall prove this theorem without assumptions on boundness and symmetry of sets under consideration.

The Kuratowski measure of noncompactness of a set G in a Banach space X is the infimum $\alpha(G)$ of those $\varepsilon > 0$ for which there is a covering of G by a finite number of sets of diameter less than ε . For a closed convex set C denote by $F(C)$ the set of all linear continuous functionals $f \in X^*$, $f \neq 0$, which are bounded above on C . For $f \in F(C)$ and $\delta > 0$ put

$$S(f, C, \delta) = \{x \in C : f(x) \geq M - \delta\}, \quad \text{where } M = \sup\{f(x) : x \in C\}.$$

To begin with, we shall give a necessary condition for the drop property which was proved in [9] for the closed unit ball.

Proposition 1. *Let C be a closed convex set with the drop property. Then C has the following property:*

$$(\alpha) \quad \alpha(S(f, C, \delta)) \rightarrow 0 \quad \text{when } \delta \rightarrow 0 \quad \text{for every } f \in F(C).$$

In particular, f attains its supremum on C .

Proof. Assume that (α) is not satisfied. Let $f \in F(C)$ be such that

$$\inf\{\alpha(S(f, C, \delta)) : \delta > 0\} > 0.$$

If there exists an $\varepsilon > 0$ such that $\alpha(S(f, C, \varepsilon)) < \infty$, i.e. $S(f, C, \varepsilon)$ is bounded, then for every $\delta > 0$ and every finite dimensional subspace L of X ,

$$\sup_{x \in S(f, C, \delta)} \left(\inf_{y \in L} \|x - y\| \right) \geq \frac{1}{2} \inf_{\nu > 0} \alpha(S(f, C, \nu)).$$

Hence, in order to obtain a contradiction, we may repeat arguments used in [9].

Suppose now that $\alpha(S(f, C, \delta)) = \infty$ for every $\delta > 0$. Choose an $x_0 \in X$ such that

$$f(x_0) > M = \sup\{f(x) : x \in C\}.$$

We shall construct a stream $\{x_0, x_1, \dots, x_n, \dots\}$ such that $f(x_n) > M$ and $\|x_i - x_j\| \geq 1$, $i \neq j$. Suppose that $\{x_0, \dots, x_n\}$ are already constructed. Take $\delta < f(x_n) - M$ and consider $S(f, C, \delta)$. Since $S(f, C, \delta)$ is unbounded, there exists an element $\bar{x}_{n+1} \in S(f, C, \delta)$ such that

$$\|\bar{x}_{n+1} - (2x_i - x_n)\| \geq 2 \quad \text{for each } i = 0, 1, \dots, n.$$

Put

$$x_{n+1} = \frac{x_n + \bar{x}_{n+1}}{2}.$$

Thus, $\|x_{n+1} - x_i\| \geq 1$ for every $i = 0, 1, \dots, n$, $x_{n+1} \in D(x_n, C)$ and $f(x_{n+1}) > M$. Let $A = \{x_0, x_1, \dots\}$. The set A is closed, disjoint with C and $x_{n+1} \in D(x_n, C)$. Hence C has not the drop property. $\square$

Lemma 2. *Let C be a closed convex set for which there is an $f \in X^*$, $f \neq 0$ such that $\lambda_1 \leq f(x) \leq \lambda_2$ on C for some reals λ_1 and λ_2 . Suppose that f “cuts C in a bounded way”, i.e. there are $x_1, x_2 \in C$ and a real λ so that $f(x_2) < \lambda < f(x_1)$ and $C \cap H_\lambda$ is bounded, where $H_\lambda = \{x \in X : f(x) = \lambda\}$. Then C is bounded.*

Proof. Observe (by a simple convexity argument) that the sets

$$C_1 = \left\{ y \in X : \lambda_1 \leq f(y) < f(x_1), \frac{f(x_1) - \lambda}{f(x_1) - f(y)}y + \frac{\lambda - f(y)}{f(x_1) - f(y)}x_1 \in C \cap H_\lambda \right\}$$

and

$$C_2 = \left\{ y \in X : f(x_2) < f(y) \leq \lambda_2, \frac{\lambda - f(x_2)}{f(y) - f(x_2)}y + \frac{f(y) - \lambda}{f(y) - f(x_2)}x_2 \in C \cap H_\lambda \right\}$$

are bounded. The relation $C \subset C_1 \cup C_2$ completes the proof. $\square$

Theorem 3. *Let C be a closed convex set in a Banach space X . If C has the drop property, then it is compact or has non-empty interior.*

Proof. If C is finite dimensional, then either C has non-empty interior or $C \subset H_\lambda = \{x \in X : f(x) = \lambda\}$ for some linear functional $f \neq 0$. If we assume that C is not compact, then C is unbounded and hence it does not have the property (α) , if C has empty interior, which is impossible by Proposition 1.

Let C be infinite dimensional. Suppose that C is noncompact and its interior is empty. The first assumption implies that there are a number $\delta > 0$ and a sequence $\{x_n\} \subset C$ such that

$$(1) \quad \text{dist}(x_{n+1}, \text{span}\{x_1, \dots, x_n\}) > \delta, \quad n = 1, 2, \dots$$

If C is bounded, then a sequence $\{x_n\}$ with Property (1) can be easily constructed for $\delta < \alpha(C)/2$.

Let C be unbounded. Suppose that $\{x_1, \dots, x_n\}$ with Property (1) are already chosen with $\delta = 1$. Assume that for every $x \in C$

$$\text{dist}(x, \text{span}\{x_1, \dots, x_n\}) \leq 1.$$

The subspace $L = \text{span}\{x_1, \dots, x_n\}$ is finite dimensional and hence there exists an $f \in X^*, \|f\| = 1$, such that

$$L \subset H = \{x \in X : f(x) = 0\}.$$

By our assumption, for every $x \in C$ there is an element $y \in L$ such that $\|x - y\| < 2$. Thus $|f(x)| < 2$. Therefore, f and $-f$ belong to $F(C)$. Let M_1 and M_2 be the supremum and infimum of f on C , respectively. Since C is unbounded, Proposition 1 implies that $M_1^2 + M_2^2 \neq 0$. Let $M_1 > 0$. By the condition (α) , there is a δ , $0 < \delta < M_1$ such that the set $C \cap \{x \in X : f(x) = M_1 - \delta\}$ is bounded. Thus, it follows from Lemma 2 that C is bounded, which contradicts our assumption.

Let $x \in X \setminus C$. Having (1), we can construct now by induction a sequence $\{y_n\}$ in the following way:

$$y_1 = \frac{x + x_1}{2} \quad \text{and} \quad y_n = \frac{y_{n-1} + x_n}{2} \quad \text{for } n \geq 2.$$

Again by induction, we obtain easily another representation of y_n :

$$(2) \quad y_n = \frac{1}{2^n}x + \sum_{i=1}^n \frac{1}{2^{n-i+1}}x_i \quad \text{for } n = 1, 2, \dots$$

By (1) and (2), we get

$$(3) \quad \|y_n - y_m\| > \frac{\delta}{2} - \frac{1}{2^n}\|x\|.$$

The sequence $\{y_n\}$ will be called a dyadic stream generated by the sequence $\{x_n\}$ with an initial point x .

Introduce a homothetic operation in X :

$$T_{z,k}(x) = z + k(x - z),$$

where $z, x \in X$, k is a real number.

Clearly, this operation maps closed (open, convex, respectively) sets onto themselves.

Write now

$$T_{\{x_1, \dots, x_n\}} = T_{x_1, 2} T_{x_2, 2} \cdots T_{x_n, 2}.$$

By an easy induction we find that $y_n \in C$ if and only if

$$x \in T_{\{x_1, \dots, x_n\}}(C).$$

Thus, the set of those x 's that the dyadic stream with an initial point x meets C , is the set

$$C_0 = \bigcup_{n=1}^{\infty} T_{\{x_1, \dots, x_n\}}(C).$$

By our assumption, C has no interior. Hence the sets $T_{\{x_1, \dots, x_n\}}(C)$ also have no interior. Therefore C_0 is of first category and there are points x such that the dyadic stream with an initial point x has no common points with C . For such a point x the set $\{y_n\}$ is, by (3), closed, disjoint with C and $y_{n+1} \in D(y_n, C)$ by construction. Hence, C has not the drop property. $\square$

It was proved in [9] that the drop property of the norm implies the reflexivity of the space. In [5] it was shown that if there exists in X a noncompact bounded closed convex set with the drop property, then X is reflexive. Theorem 3 leads to the following question.

Problem. Let C be a closed unbounded convex set in a Banach space X . Does the drop property of C imply that X is reflexive?

Proposition 4. *Let C be a closed convex symmetric set, having the drop property. Then C is bounded.*

Proof. Let $x \notin C$. Then there is a functional $f \neq 0$ which separates x and C . Let $f(x) \geq M = \sup\{f(x) : x \in C\}$. Since C is symmetric, then $\inf\{f(x) : x \in C\} = -M$. If C is noncompact, Proposition 1 implies that $M > 0$. Then, applying Proposition 1 once more, we get that f cuts C in a bounded way. Thus, Lemma 2 implies that C is bounded. $\square$

Clearly, when C is a bounded closed convex set, $D(b, C)$ is closed for each $b \notin C$. If C is unbounded, this is not always true. We shall show that the drop property coincides with the “closed” drop property, where in the definition we take a closed drop $\overline{D(b, C)}$.

Proposition 5. *Let C be a closed convex (unbounded) set such that $\text{int } C \neq \emptyset$ and C has the property (α) . Then for every $b \notin C$ the set $D(b, C)$ is closed.*

Proof. Assume that $D(b, C)$ is not closed, i.e. there is a point $b \notin C$ such that there exists an $a \in \overline{D(b, C)} \setminus D(b, C)$. Thus, there are $y_n \in C$ and $0 \leq \lambda_n \leq 1$ such that

$$x_n = \lambda_n b + (1 - \lambda_n) y_n \rightarrow a.$$

Without loss of generality we may suppose that the sequence $\{\lambda_n\}$ is convergent. It is easy to see that $\lambda_n \rightarrow 1$ and $\|y_n\| \rightarrow \infty$. Then by a simple convexity argument we obtain that the ray $r = \{b + \eta(a - b) : \eta \geq 1\}$ is also contained in the set $\overline{D(b, C)} \setminus D(b, C)$. Since $r \cap C = \emptyset$ and $\text{int } C \neq \emptyset$, there exists an $f \in X^*$, $\|f\| = 1$ such that

$$M = \sup\{f(x) : x \in C\} \leq \inf\{f(x) : x \in r\}.$$

It is easy to verify that $f(a) = f(b)$, i.e. $r \subset H = \{x \in X : f(x) = f(a)\}$. We have that $M \leq f(a)$.

By property (α) , there is a $\delta > 0$ such that $\alpha(S(f, C, \delta)) < 1$ which implies that $S(f, C, \delta)$ is bounded. Write

$$H^1 = \{x \in X : f(x) = M - \delta\} \quad \text{and} \quad K_1 = C \cap H^1.$$

Hence, $q_1 = \text{diam}(K_1) < \infty$.

In order to consider together the cases $M = f(a)$ and $M < f(a)$, take a point $y \in C$ such that $M - \delta < f(y) < f(a)$. Write:

$$d = f(y) - M + \delta.$$

Choose an arbitrary element $e \in X$ with $f(e) = 1$. We will take into account projections in the direction e on the hyperplane defined by f . Denote by p' the projection of y on H^1 , by p_n the projections of y_n on H . Since $S(f, C, \delta)$ is bounded and $\|y_n\| \rightarrow \infty$, then for n great enough we get $f(y_n) < M - \delta$, whence the segment $[y, y_n] \subset C$ crosses K_1 . Denote this common point by y_n'' . Let y_n'' be the common point of H and the ray containing y with the initial point y_n . Observe that the set $\{y_n''\}$ is bounded:

$$\text{diam}(\{y_n''\}) \leq q_1 \frac{f(a) - f(y)}{d} = q.$$

Since $\|p' - y\| \geq |f((p' - y))| = d$, we have

$$\frac{\|p_n - y_n''\|}{\|p_n - y_n\|} = \frac{\|p' - y_n'\|}{\|p' - y\|} \leq \frac{\|p' - y_n'\|}{d} \leq \frac{q_1 + \|p'\|}{d}.$$

Therefore,

$$\frac{\|p_n - b\|}{\|p_n - y_n\|} \leq \frac{\|p_n - y_n''\|}{\|p_n - y_n\|} + \frac{\|y_n'' - b\|}{\|p_n - y_n\|} \leq \frac{q_1 + \|p'\|}{d} + \frac{q + \|b\|}{d} = q_2.$$

Let x_n' be the projection of x_n on H . Then,

$$(4) \quad \frac{\|x_n' - b\|}{\|x_n' - x_n\|} = \frac{\|p_n - b\|}{\|p_n - y_n\|} \leq q_2 < \infty.$$

On the other hand, $\|x_n - a\| \rightarrow 0$, implies $f(x_n) \rightarrow f(a)$. Since

$$x_n' - x_n = (f(a) - f(x_n))e,$$

we get $\|x_n' - x_n\| \rightarrow 0$. Moreover,

$$\|x_n' - b\| \geq \|b - a\| - \|x_n - a\| - \|x_n - x_n'\| \rightarrow \|b - a\| \neq 0.$$

Thus,

$$\frac{\|x_n' - b\|}{\|x_n' - x_n\|} \rightarrow \infty,$$

which contradicts (4). Therefore, $\overline{D(b, C)} = D(b, C)$. $\square$

Corollary 6. *Let C be a closed convex (unbounded) set in a Banach space. Then C has the drop property if and only if C has the closed drop property.*

Proof. Clearly, the closed drop property implies the drop property. The converse implication follows immediately from Proposition 1, Theorem 3 and Proposition 5. $\square$

Montesinos [6] has obtained a nice characterization of the drop property of the norm. A similar result has been given in [5] for bounded sets with non-empty interior. Combining the statement from [5] and Theorem 3, we have

Theorem 7. *Let C be a bounded noncompact closed convex set in a (necessarily reflexive) Banach space X . The following conditions are equivalent:*

- (i) C has the drop property;
- (ii) $\text{int } C \neq \emptyset$ and C has the property (α) (here $F(C) = X^* \setminus \{0\}$);
- (iii) $\text{int } C \neq \emptyset$, C is weakly compact and every support point x of C is a point of continuity (i.e. $\|y_n - x\| \rightarrow 0$ whenever $\{y_n\} \subset C$, y_n tending weakly to x).

When C is not bounded we do not know whether the drop property of C implies the reflexivity of X . In reflexive Banach spaces we can characterize the drop property by (ii), as well. Here we have to overcome some difficulties since a general drop theorem of the Danes type (i.e. a theorem for a closed set A of a positive distance from C) does not hold.

Theorem 8. *Let C be an unbounded closed convex set in a reflexive Banach space $(X, \|\cdot\|)$. The following conditions are equivalent:*

- (i) C has the drop property;
- (ii) $\text{int } C \neq \emptyset$ and C has the property (α) .

Clearly, the property (α) implies that every support point of C is a point of continuity.

Proof. (i) $\Rightarrow$ (ii) follows from Proposition 1 and Theorem 3.

(ii) $\Rightarrow$ (i). Since X is reflexive, by the theorem of Montesinos [6], X has an equivalent norm having the drop property. The drop property of C and the property (α) are invariant under isomorphisms. We therefore can assume without loss of generality that the original norm $\|\cdot\|$ (i.e. the closed unit ball B in $(X, \|\cdot\|)$) has the drop property.

Clearly, for every $x \notin C$ we get

$$(5) \quad \text{dist}(a, C) < \text{dist}(x, C) \quad \text{whenever } a \in D(x, C), \ a \neq x.$$

Assume that C has not the drop property, i.e. there is a closed set A , disjoint with C , such that

$$(6) \quad \text{for every } x \in A \text{ there is a point } a \in D(x, C) \cap A, \ a \neq x.$$

Let x_1 be an arbitrary point of A . Choose $\{x_1, \dots, x_n\}$ and put

$$d'_n = \inf\{\text{dist}(a, C) : a \in A \cap D(x_n, C)\}.$$

By (6), d'_n is well-defined. Select an $x_{n+1} \in D(x_n, C)$, $x_{n+1} \neq x_n$, so that

$$(7) \quad d_{n+1} = \text{dist}(x_{n+1}, C) < d'_n + \frac{1}{n}.$$

By (5), $\{d_n\}$ is convergent. Write $\varepsilon = \lim_n d_n$. We have two possibilities.

(i) $\varepsilon = 0$. Then, there are $y_n \in C$ such that $\|x_n - y_n\| \rightarrow 0$.

Let $A_1 = \text{co}\{x_n\}$. Since $\{x_n\}$ is a stream, by the result of Montesinos [6], we get $A_1 \cap C = \emptyset$. Since $\text{int } C \neq \emptyset$, this implies that there is an $f \in X^*$, $f \neq 0$, which separates A_1 and C . Suppose that $f \in F(C)$ and put $M = \sup\{f(x) : x \in C\}$. Clearly,

$$f(y_n) \rightarrow M.$$

Hence, by the property (α) , $\{y_n\}$ is bounded. Since X is reflexive, without loss of generality, we can assume that $\{y_n\}$ is weakly convergent to a point $y \in C$. Thus $f(y) = M$ and by (α) , $\|y_n - y\| \rightarrow 0$. Hence $\|x_n - y\| \rightarrow 0$. Since A is closed, we get

$$y \in A \cap C,$$

which contradicts our assumption.

(ii) $\varepsilon > 0$. Let

$$C_1 = C + \varepsilon B.$$

Clearly, C is weakly closed, εB is weakly compact. Thus, C_1 is closed. We shall show that C_1 has the property (α) . Let $f \in F(C_1)$. Hence $f \in F(C)$. It is easy to observe that

$$(8) \quad S(f, C_1, \delta) \subset S(f, C, \delta) + S(f, \varepsilon B, \delta) \quad \text{for every } \delta > 0.$$

Since εB has the drop property and C has the property (α) , we obtain by properties of the Kuratowski measure of noncompactness and (8) that

$$\alpha(S(f, C_1, \delta)) \leq \alpha(S(f, C, \delta)) + \alpha(S(f, \varepsilon B, \delta)) \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

By (5), we have that $d_n \downarrow \varepsilon$. Thus $x_n \notin C_1$ and $\text{dist}(x_n, C_1) \downarrow 0$. Clearly, $x_{n+1} \in D(x_n, C_1)$. As in point (i) we prove that $\{x_n\}$ has a subsequence (denoted again by $\{x_n\}$) which is convergent in the norm topology to an element x . It is easy to see that this subsequence has the same required properties. The set A is closed, hence $x \in A$. Then, by (6), there is an $a \in D(x, C) \cap A$, $a \neq x$.

By Proposition 5, $D(b, C)$ is closed for every $b \notin C$. Since $x_n \rightarrow x$ and $D(x_i, C) \supset \{x_{i+1}, x_{i+2}, \dots\}$, we get

$$x \in D(x_i, C) \quad \text{for } i = 1, 2, \dots$$

We have $\text{dist}(x, C) = \varepsilon$, by (5), $\text{dist}(a, C) < \text{dist}(x, C)$. We therefore can choose an integer n such that

$$\frac{1}{n} < \text{dist}(x, C) - \text{dist}(a, C).$$

Hence

$$\text{dist}(x_{n+1}, C) > \text{dist}(a, C) + \frac{1}{n} \geq d'_n + \frac{1}{n},$$

which contradicts (7).

Finally, we conclude that C has the drop property. $\square$

Theorem 9. *Let C_1 and C_2 be closed convex sets with the drop property in a reflexive Banach space X . Then, the intersection, convex hull and linear combination of bounded sets having the drop property has again the drop property, i.e.*

- (i) *if $C^1 = C_1 \cap C_2$ is bounded, then C^1 has the drop property;*
- (ii) *if C_1 and C_2 are bounded, then $C^2 = \text{co}(C_1, C_2)$ has the drop property;*
- (iii) *if C_1 is bounded, then $C^3 = \lambda C_1 + \mu C_2$, $\lambda, \mu \neq 0$, has the drop property.*

Proof. (i) Let $C^1 = C_1 \cap C_2$. If $\text{int } C_1 = \emptyset$, then it follows from Theorem 3 and Proposition 1 that C^1 is compact. Hence C has the drop property. Let $\text{int } C \neq \emptyset$. Let x_0 be a boundary point of C^1 . Then x_0 is a boundary point of C_1 or of C_2 . Suppose that $x_0 \in C_1$. Since $\text{int } C_1 \neq \emptyset$, x_0 is a support point of C_1 . Then by the drop property of C_1 , x_0 is a point of continuity for C_1 , hence also for C^1 .

By assumption, C^1 is a bounded closed convex set in a reflexive space, hence C^1 is weakly compact. Then by Theorem 7, C^1 has the drop property.

(ii) The case when C_1 and C_2 are compact is trivial. According to Theorem 3, at least one of the sets C_1 and C_2 , has a non-empty interior. Hence $\text{int } C^2 \neq \emptyset$, where $C^2 = \text{co}(C_1, C_2)$. The sets C_1 and C_2 are weakly compact. Thus C^2 is also weakly compact. Let z_0 be a support point of C^2 and $\{z_n\} \subset C^2$ be a sequence weakly convergent to z_0 . We shall prove that:

Starting from an arbitrary subsequence of $\{z_n\}$ (denoted again by $\{z_n\}$), we can choose a new subsequence strongly converging to z_0 . Let

$$z_n = \eta_n x_n + (1 - \eta_n) y_n, \quad \text{where } 0 \leq \eta_n \leq 1, x_n \in C_1 \text{ and } y_n \in C_2.$$

Since C_1 and C_2 are weakly compact, we can take a subsequence such that $\eta_n \rightarrow \eta$, $\{x_n\}$ and $\{y_n\}$ converge weakly to $x_0 \in C_1$ and $y_0 \in C_2$, respectively. Since C_1 and C_2 are bounded, it is enough to consider the case

$$z_n = \eta x_n + (1 - \eta)y_n.$$

If η is equal to either 1 or 0, then z_0 is a support point of C_1 or C_2 . If $0 < \eta < 1$, then by convexity argument, x_0 and y_0 are support points of C_1 and C_2 , respectively. Since C_1 and C_2 have the drop property, we find $x_n \rightarrow x_0$ and $y_n \rightarrow y_0$. Hence $z_n \rightarrow z_0$. Theorem 7 completes the proof. If C_1 is the closed unit ball B and $C_2 = \{x, -x\}$ for $x \notin B$, then this statement was proved in [10].

(iii) Without loss of generality, suppose that $\lambda, \mu > 0$. The set C_1 is weakly compact, thus $C^3 = \lambda C_1 + \mu C_2$ is closed. If C_1 and C_2 are compact, then our conclusion is immediate. If it is not so, then $\text{int } C^3 \neq \emptyset$. Let $f \in F(C^3)$. Hence, $f \in F(C_2)$, by the boundness of C_1 . It is easy to see that

$$S(f, C^3, \delta) \subset S\left(f, C_1, \frac{\delta}{\lambda}\right) + S\left(f, C_2, \frac{\delta}{\mu}\right).$$

Thus, by Proposition 1,

$$\alpha(S(f, C^3, \delta)) \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Then from Theorem 8 it follows that C^3 has the drop property. $\square$

Now, we shall give an example of an unbounded set having the drop property. We shall use the following

Proposition 10. *Let C be a closed convex set with non-empty interior in a reflexive Banach space. If every support point of C is a point of continuity and for every $x \notin C$ the set $D(x, C) \setminus C$ is bounded, then C has the drop property.*

Proof. By our assumption, we get that for each $x \notin C$, the drop $D(x, C)$ is closed. Take an arbitrary closed set A disjoint with C . Fix a point $a \in A$. Then the set $A_1 = A \cap D(a, C)$ is closed. Let B be the closed unit ball. Without loss of generality, we can assume that B has the drop property [6].

For n great enough we have

$$(9) \quad nB \supset D(a, C) \setminus C \quad \text{and} \quad \text{int}(nB \cap C) \neq \emptyset.$$

Fix an n satisfying (9) and write:

$$C_1 = nB \cap C.$$

C_1 is weakly compact and every support point of C_1 is a support point either of nB or of C and hence, it is a point of continuity for C_1 . Therefore, C_1 has the drop property. A_1 is closed and $A_1 \cap C_1 = \emptyset$. Thus, there is a point $b \in A_1$ such that

$$A_1 \cap D(b, C_1) = \{b\}.$$

By this and (9) we obtain

$$A \cap D(b, C) = \{b\}. \quad \square$$

Example 1. Let $(X, \|\cdot\|)$ be a reflexive Banach space and $\|\cdot\|$ have the drop property. Let g be a differentiable convex function defined for $u \in [0, +\infty)$ such that g' is strictly increasing and $g'(0) \geq 0$. Assume that

$$\lim_{u \rightarrow \infty} \frac{g(u)}{u} = +\infty.$$

Then the set

$$\Omega = \{(x, t) \in X \times \mathbb{R} : t \geq g(\|x\|)\} \subset X \times \mathbb{R}.$$

has the drop property.

By easy computation, we get that for every $(x_0, t_0) \in X \times \mathbb{R}$ the set $D((x_0, t_0), \Omega) \setminus \Omega$ is bounded and every support point of Ω is a point of continuity. Then we may apply the above proposition.

Observe that the set Ω , defined in Example 1, contains a ray. This property holds in general in reflexive Banach spaces for unbounded sets having the drop property.

Proposition 11. *Let $(X, \|\cdot\|)$ be a reflexive Banach space. Let C be a closed convex unbounded set having the drop property. Then C contains a ray.*

The proof is based on the following

Lemma 12. *Let C be a closed convex unbounded set in a reflexive Banach space X . Suppose that there is a closed hyperplane $H_\lambda = \{x \in X : f(x) = \lambda\}$ which cuts C , i.e. there are $x_1, x_2 \in C$ such that $f(x_2) < \lambda < f(x_1)$ and $H_\lambda \cap C$ is bounded. Then C contains a ray.*

Proof. It follows from Lemma 2 that either

$$(10) \quad \sup\{f(x) : x \in C\} = +\infty,$$

or

$$(11) \quad \inf\{f(x) : x \in C\} = -\infty.$$

Suppose that (10) holds. Let $z \in H_\lambda \cap C$. Write:

$$r(z) = \sup\{f(x_2 + t(z - x_2)) : t \geq 1, x_2 + t(z - x_2) \in C\}.$$

Clearly by (10),

$$\sup\{r(z) : z \in C \cap H_\lambda\} = +\infty.$$

By the reflexivity of the space X , we can find a sequence $\{z_n\} \subset C \cap H_\lambda$ such that $r(z_n) > n$ and $\{z_n\}$ weakly tends to a point $z_0 \in C \cap H_\lambda$. It is well-known

that for any sequence $\{z_n\}$ tending weakly to z_0 there is a sequence of its convex combinations tending to z_0 in the norm. Thus there is a sequence of integers $1 = p_1 \leq q_1 < p_2 \leq q_2 < \dots$ and coefficients $n_i \geq 0$ such that

$$\sum_{i=p_n}^{q_n} \eta_i = 1, \quad y_n = \sum_{i=p_n}^{q_n} \eta_i z_i \quad \text{for every } n$$

and y_n tends to z_0 in the norm topology. It is easy to observe that

$$r(y_n) \geq \inf_{p_n \leq i \leq q_n} r(z_i).$$

Hence

$$\lim_{n \rightarrow \infty} r(y_n) = +\infty.$$

This implies that $r(z_0) = +\infty$, i.e. C contains a ray. $\square$

Proposition 11 is an immediate consequence of Proposition 1 and Lemma 12.

Remark. Without the reflexivity, Lemma 12 does not hold, as it is shown by the following

Example 2. Let $X = l^1$. Let $C = \{x = (x_n) \in l^1 : 0 \leq x_i \leq 1 \text{ for every } i \in \mathbb{N}\}$. It is easy to verify that C does not contain a ray. Let $f(x) = x_1 + x_2 + \dots$. Write $H_1 = \{x \in X : f(x) = 1\}$. Observe that $f(x) = \|x\|$ for $x \in C$. Thus, $H_1 \cap C$ is bounded. Clearly, H_1 cuts C , since $\inf\{f(x) : x \in C\} = 0$ and $\sup\{f(x) : x \in C\} = +\infty$.

Lemma 12 can be generalized by induction for subspaces of codimension k in the following way:

Lemma 12'. *Let C be a closed convex unbounded set in a reflexive Banach space. Let H be an affine closed subset of codimension k . We assume that H cuts C , i.e. for every linear continuous functional f , constant on H , we have*

$$\inf\{f(x) : x \in C\} < f(H) < \sup\{f(x) : x \in C\}.$$

If $H \cap C$ is bounded then C contains a ray.

Proof. There are a finite system of linear continuous functionals $f_1, \dots, f_k$ and numbers $a_1, \dots, a_k$ such that

$$H = \{x \in X : f_i(x) = a_i, i = 1, \dots, k\}.$$

Let

$$H_1 = \{x \in C : f_i(x) = a_i, i = 2, \dots, k\}.$$

By Lemma 12, H_1 either contains a ray or is bounded. In the first case the proof is finished. In the second one we repeat our argument for the set

$$H_2 = \{x \in C : f_i(x) = a_i, i = 3, \dots, k\}.$$

After at most k steps we conclude that C contains a ray. $\square$

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