On nearly uniformly convex sets

by

D. N. KUTZAROVA and S. ROLEWICZ

The notion of nearly uniformly convex Banach spaces was introduced by Huff [2] and independently by Goebel and Sekowski [1]. In [7] it was shown that in a certain way it is a uniformization of drop property for norms.

In [3] the authors considered drop property for convex sets. In the present paper they investigate nearly uniformly convex sets which need not be balls in Banach spaces.

Let $(X, \|\cdot\|)$ be a real Banach space. The norm (or the closed unit ball B) is said to be nearly uniformly convex (NUC) [2] if one of the following conditions is satisfied:

Either

- (i) for every $\varepsilon > 0$ there exists a δ , $0 < \delta < 1$ such that for every sequence $\{x_n\} \subset B$ with the property

$$\text{sep}(x_n) := \{\inf \|x_n - x_m\| : n \neq m\} > \varepsilon$$

we have

$$\text{co}(x_n) \cap (1 - \delta)B \neq \emptyset;$$

or (equivalently, [1], [7]);

- (ii) for every $\varepsilon > 0$ there is a δ , $0 < \delta < 1$ such that for every convex set $E \subset B$ with $\alpha(E) > \varepsilon$, we have

$$E \cap (1 - \delta)B \neq \emptyset,$$

where $\alpha(E)$, denotes the index of noncompactness of Kuratowski (see below).

Recall the following notions. Let E be an arbitrary set contained in X . By the index of noncompactness of Kuratowski $\alpha(E)$ we shall denote the infimum of those ε for which there is a finite decomposition of E by sets $E_1, \dots, E_n$, $E = \bigcup_{i=1}^n E_i$, such that the diameters of all sets E_i , is non-greater than ε .

Let f be a continuous linear functional of norm one. We set

$$S(f, B, \delta) = \{x : \|x\| \leq 1, f(x) \geq 1 - \delta\}.$$

Having these notions we can give the following characterizations of nearly uniform convexity.

Proposition 1. *Let $(X, \|\cdot\|)$ be a Banach space.*

- (i) ([7]) $\|\cdot\|$ is (NUC) if and only if B has the uniform property (α) , i.e. for every $\varepsilon > 0$ there is a δ , $0 < \delta < 1$, such that $\alpha(S(f, B, \delta)) \leq \varepsilon$ for every continuous linear functional f of norm one;
- (ii) ([2]) $\|\cdot\|$ is (NUC) if and only if X is reflexive and $\|\cdot\|$ is uniformly Kadec-Klee (UKK) (i.e. for every $\varepsilon > 0$ there is a δ , $0 < \delta < 1$, such that $\{x_n\} \subset B$, $x_n \rightarrow x$ weakly and $\text{sep}(x_n) > \varepsilon$ imply $x \in (1 - \delta)B$).

Recall that property (α)

- (α) for each linear continuous functional f and each $\varepsilon > 0$ there is a δ , $0 < \delta < 1$ such that $\alpha(S(f, B, \delta)) \leq \varepsilon$

is equivalent to the fact that the norm $\|\cdot\|$ has the drop property i.e. for each closed set C disjoint with the unit ball B there is a point $a \in C$ such that

$$(1) \quad C \cap \text{conv}(\{a\} \cup B) = \{a\}$$

(see [6]).

Therefore each uniformly convex norm has the drop property [7]. There are reflexive Banach spaces, in which no equivalent nearly uniformly convex norm exists [2, 7].

In paper [3] the authors extended the notion of drop property to arbitrary closed convex sets. In the present paper we shall extend the notion (NUC) to arbitrary closed convex sets. There are some difficulties connected with the fact that (NUC) depends on the point with respect to which it is regarded.

Definition 2. Let C be a closed convex set in a real Banach space X . We say that C is (NUC) with respect to a center $a \in C$ if either

- (i) for every $\varepsilon > 0$ there is a δ , $0 < \delta < 1$, such that for every sequence $\{x_n\} \subset C$ with $\text{sep}(x_n) > \varepsilon$, we have

$$(2) \quad \text{co}(x_n) \cap (a + (1 - \delta)(C - a)) \neq \emptyset;$$

or (equivalently)

- (ii) for every $\varepsilon > 0$ there is a δ , $0 < \delta < 1$, such that for every convex set $E \subset C$ with $\alpha(E) > \varepsilon$, we have

$$(3) \quad E \cap (a + (1 - \delta)(C - a)) \neq \emptyset.$$

We shall give two simple examples which show that it is not possible to generalize some statements for a (NUC) norm.

Example 3. Let $X = l^2$ and let B be the closed unit ball. Let

$$C_1 = \{x = (x_n) : x \in B, x_1 \geq 0\}.$$

Then C_1 is (NUC) with respect to the origin. However, for the functional f defined by $f(x) \rightarrow -x_1$ we have

$$(4) \quad \inf\{\alpha(S(f, C_1, \delta)) : \delta > 0\} > 0.$$

Hence, C_1 fails the drop property.

Evidently, C_1 has non-empty interior. By (4) and Proposition 5 below, we conclude that C_1 is not (NUC) with respect to any interior point.

Example 4. Let X be as in Example 3. Write:

$$C_2 = \{x \in B : x_1 = 0\}.$$

Clearly, C_2 is (NUC) with respect to the origin. Hence Theorem 3 of [3] does not hold for (NUC), because C_2 is noncompact and has no interior points.

If we consider (NUC) with respect to an interior point, then we prove a similar characterization by the uniform property (α) as in [7] (cited here as Proposition 1 (i)).

Let $F(C) = \{f \in X^* : \sup_{x \in C} f(x) < +\infty\}$. Let $f \in F(C)$. Denote by M the supremum of f on C . For $a \in C$ and $0 < \delta < 1$ let

$$\begin{aligned} S_1(f, C, a, \delta) &= \{x \in C : f(x) \geq f(a) + (1 - \delta)(M - f(a))\} \\ &= \{x \in C : f(x) \geq M - \delta(M - f(a))\}. \end{aligned}$$

Proposition 5. *Let C be a closed convex set with non-empty interior in a Banach space $(X, \|\cdot\|)$.*

- (i) *If C is (NUC) with respect to a center $a \in \text{int } C$ then $\alpha(S_1(f, C, a, \delta)) \rightarrow 0$ as $\delta \rightarrow 0$ uniformly for $f \in F(C)$;*
- (ii) *if $a \in C$ (a is not necessarily an interior point) and $\alpha(S_1(f, C, a, \delta)) \rightarrow 0$ as $\delta \rightarrow 0$ uniformly for $f \in F(C)$, then C is (NUC) with center a ;*
- (iii) *if $a \in \text{int } C$, $b \in C$ and $\alpha(S_1(f, C, a, \delta)) \rightarrow 0$ as $\delta \rightarrow 0$ uniformly for $f \in F(C)$, then $\alpha(S_1(f, C, b, \delta)) \rightarrow 0$ as $\delta \rightarrow 0$ uniformly for $f \in F(C)$.*

Proof. (i) Let ε be an arbitrary positive number. Choose a δ_1 , $0 < \delta_1 < 1$, in such a way that for every convex set $E \subset C$, such that $\alpha(E) > \varepsilon$, we have $E \cap (a + (1 - \delta_1)(C - a)) \neq \emptyset$.

Put $\delta = \delta_1/2$. Let $f \in F(C)$ and let M be its supremum on C . Take an $x \in a + (1 - \delta_1)(C - a)$. Then

$$f(x) \leq M - 2\delta(M - f(a)).$$

Since $a \in \text{int } C$, we find

$$(5) \quad M - f(a) > 0.$$

Hence

$$f(x) < M - \delta(M - f(a)).$$

This implies

$$S_1(f, C, a, \delta) \cap (a + (1 - \delta_1)(C - a)) = \emptyset.$$

Hence

$$\alpha(S_1(f, C, a, \delta)) \leq \varepsilon.$$

(ii) Let ε be an arbitrary positive number. Choose $\delta, 0 < \delta < 1$, so that for each $f \in F(C)$,

$$\alpha_1(S, (f, C, a, \delta)) \leq \varepsilon.$$

Let E be a convex subset of C with $\alpha(E) > \varepsilon$. Suppose

$$E \cap (a + (1 - \delta)(C - a)) = \emptyset.$$

Since $\text{int } C \neq \emptyset$, there exists a functional $f \in X^*$, $\|f\| = 1$, such that

$$\sup\{f(x) : x \in a + (1 - \delta)(C - a)\} \leq \inf\{f(x) : x \in E\}.$$

Observe that $f \in F(C)$ and denote by M its supremum on C . Then for every $x \in E$ we get

$$f(x) \geq f(a) + (1 - \delta)(M - f(a)),$$

i.e.

$$E \subset S_1(f, C, a, \delta).$$

Thus $\alpha(E) \leq \varepsilon$, which contradicts our assumption.

(iii) Since $a \in \text{int } C$, there are a $d \in C$ and $0 < \eta < 1$ such that $a - b = (1 - \eta)(d - b)$. Hence by the convexity of C ,

$$a - b + \eta(C - b) \subset C - b.$$

Therefore, for every $f \in F(C)$ with $M = \sup\{f(x) : x \in C\}$, we get

$$f(a - b) \leq (1 - \eta)(M - f(b)).$$

Let $\varepsilon > 0$. By the assumption, there is a $\delta > 0$ such that $\alpha(S_1(f, C, a, \delta)) \leq \varepsilon$.

Let $\delta_1 = \eta\delta$. If $x \in S_1(f, C, b, \delta_1)$, then

$$\begin{aligned} f(x) &\geq M - \delta\eta(M - f(b)) = M - \delta(M - f(b)) + \delta(1 - \eta)(M - f(b)) \\ &\geq M - \delta(M - f(b)) + \delta f(a - b) = M - \delta(M - f(a)). \end{aligned}$$

Thus,

$$S_1(f, C, b, \delta_1) \subset S_1(f, C, a, \delta),$$

whence $\alpha(S_1(f, C, b, \delta_1)) \leq \varepsilon$. □

Remark 6. It is easy to see from the proof of Proposition 5 (i) that if C (not necessarily with a non-empty interior) is (NUC) with an arbitrary center $a \in C$, then every functional $f \in F(C)$ attains its supremum M on C . Indeed, let $f \in F(C)$. Either $f(a) = M$, hence the conclusion is obvious, or (5) is fulfilled. Then by the proof of (i), $\alpha(S_1(f, C, a, \delta)) \rightarrow 0$ as $\delta \rightarrow 0$. This implies that f attains its supremum on C .

Corollary 7. *If a closed convex set C is (NUC) with respect to a point $a \in \text{int } C$, then C is (NUC) with an arbitrary center $b \in C$.*

If C is bounded we may use the usual slices $S(f, C, \delta)$.

Corollary 8. *Let C be a bounded closed convex set. Then C is (NUC) with respect to any interior point if and only if*

$$\alpha(S(f, C, \delta)) \rightarrow 0 \quad \text{as } \delta \rightarrow 0$$

uniformly for $f \in X^$, $\|f\| = 1$.*

Corollary 9. *Let C be a bounded closed convex set in a Banach space X with non-empty interior. If C is (NUC) with a center $a \in \text{int } C$, then C has the drop property. In particular, by [4], X is reflexive.*

Proposition 10. *Let C be a closed convex unbounded set in a finite dimensional space X . If C has the drop property, then C is (NUC) with respect to any point $a \in C$.*

Proof. By Theorem 3 of [3], $\text{int } C \neq \emptyset$. Let $f \in F(C)$. By Proposition 1 of [3], there is $\delta > 0$ such that $S(f, C, \delta)$ is bounded. Thus, by convexity, all slices of f are bounded. Since the space X is finite dimensional, all bounded sets have the measure of noncompactness equal to zero. This, and Proposition 5, together imply that C is (NUC). □

Every closed convex cone in a Banach space is (NUC) with respect to its vertex.

Problem 11. Let C be a closed convex set in an infinite dimensional Banach space. Does (NUC) of C with respect to an $a \in \text{int } C$ imply boundedness of C ?

In the sequel, we shall consider bounded sets only.

We shall say that a convex set C satisfies (NUC') with respect to a center $a \in C$ if it satisfies Definition 2 with the conditions (2) resp. (3) replaced by (6) resp. (7). Here

$$(6) \quad \overline{\text{co}}(x_n) \cap (a + (1 - \delta)(C - a)) \neq \emptyset$$

and

$$(7) \quad \overline{E} \cap (a + (1 - \delta)(C - a)) \neq \emptyset.$$

Clearly, if C is (NUC) with a center a , then it is also (NUC') with the same center. If the center a is an interior point of C then the notions (NUC) and (NUC') with respect to a are equivalent.

Problem 12. Do the notions (NUC) and (NUC') coincide if the center is not an interior point of C ?

For bounded C , condition (NUC') may be recast in a way similar to that of Huff [5], given as Proposition 1 (ii) here.

Theorem 13. *Let C be a bounded closed convex set in a Banach space X and let $a \in C$. Then, C is (NUC') with respect to a center a if and only if C is weakly compact and is (UKK) with respect to a (i.e. for every $\varepsilon > 0$ there is a δ , $0 < \delta < 1$, such that $\{x_n\} \subset C$, $x_n \rightarrow x$ weakly and $\text{sep}(x_n) > \varepsilon$ imply $x \in a + (1 - \delta)C$).*

Proof. By linearity, we can assume without loss of generality that $a = 0$.

Sufficiency. Let $\varepsilon > 0$. Choose a $\delta > 0$ according to (UKK). Let $\{x_n\} \subset C$ with $\text{sep}(x_n) > \varepsilon$. Since C is weakly compact, we can select a subsequence of $\{x_n\}$, which weakly converges to some $x \in C$. Hence $x \in \overline{\text{co}}(x_n)$. By (UKK), $x \in (1 - \delta)C$, i.e. $\overline{\text{co}}(x_n) \cap (1 - \delta)C \neq \emptyset$.

Necessity. Let $\varepsilon > 0$. Choose a δ , $0 < \delta < 1$, corresponding to ε in the definition of (NUC') in the form (6). Let $\{x_n\} \subset C$ with $\text{sep}(x_n) > \varepsilon$. Then we can find by induction a sequence of integers $1 = p_1 \leq q_1 < p_2 \leq q_2 < \dots$, coefficients $\lambda_i \geq 0$, and elements $y_n, z_n \in C$ such that for every n we have $z_n \in (1 - \delta)C$, $\|y_n - z_n\| < 1/n$ and

$$\sum_{i=p_n}^{q_n} \lambda_i = 1, \quad y_n = \sum_{i=p_n}^{q_n} \lambda_i x_i.$$

If $\{x_n\}$ tends weakly to x , then so does $\{y_n\}$. Hence $\{z_n\}$ also tends weakly to x . Since C is convex and closed, hence weakly closed, we get $x \in (1 - \delta)C$. Therefore C is (UKK).

It remains to show that C is weakly compact. Since the slices are closed, it follows from Remark 6 that every functional $f \in X^*$ attains its supremum on C . Thus, by the James theorem, C is weakly compact. $\square$

Theorem 14. *Let X be a (reflexive) Banach space. Let C_1 and C_2 , be bounded, closed convex sets which are (NUC) with respect to $a \in \text{int } C_1$ and $b \in \text{int } C_2$, respectively.*

- (i) *If $C^1 = C_1 \cap C_2 \neq \emptyset$, then C^1 is (NUC) (i.e. (NUC) with respect to any point from the set);*
- (ii) *$C^2 = \lambda C_1 + \mu C_2$, $\lambda, \mu \neq 0$, is (NUC);*
- (iii) *$C^3 = \text{co}(C_1, C_2)$ is (NUC);*
- (iv) *$C^4 = \text{co}(C_1, K)$ for K compact, is (NUC).*

Proof. (i) If C^1 has no interior points, then by Theorem 3 of [3] C^1 is compact. Otherwise, take a $d \in \text{int } C^1$. Then, by Corollary 7, C_1 and C_2 are (NUC) with center d . By Theorem 13, C^1 is (NUC) with center d . Thus, by Corollary 7, C^1 is (NUC) with respect to any point from C^1 .

(ii) Without loss of the generality, we may assume that $\lambda, \mu > 0$. Let $\varepsilon > 0$. According to Corollary 9, choose $\delta_1 = \delta_1(\varepsilon) > 0$ and $\delta_2 = \delta_2(\varepsilon) > 0$ so that $\alpha(S(f, C_1, \delta_1)) \leq \varepsilon$ and $\alpha(S(f, C_2, \delta_2)) \leq \varepsilon$ for every $f \in X^*$, $\|f\| = 1$.

Write $\delta = \min(\lambda\delta_1, \mu\delta_2)$. Observe that

$$S(f, C^2, \delta) \subset S\left(f, C_1, \frac{\delta}{\lambda}\right) + S\left(f, C_2, \frac{\delta}{\mu}\right) \subset S(f, C_1, \delta_1) + S(f, C_2, \delta_2).$$

Therefore

$$\alpha(S(f, C^2, \delta)) \leq 2\varepsilon.$$

This, Corollary 7 and Corollary 8 together imply (ii).

The proof of (iii) is based on the following simple auxiliary results.

Lemma 15. *Let C be a bounded closed convex set and let $a, b \in \text{int } C$. Then for every $0 < \delta < 1$ there exists a δ_1 , $0 < \delta_1 < 1$, dependent on δ , C , a and b such that $x \in b + (1 - \delta)(C - b)$ implies $x \in a + (1 - \delta)(C - a)$.*

Proof. Since C is bounded and $a, b \in \text{int } C$, there exists an η , $0 < \eta < 1$ such that

$$\eta(C - a) \subset C - b.$$

By $x \in b + (1 - \delta)(C - b)$ and the convexity of C ,

$$x + \delta(C - b) \subset C.$$

Hence $x - a + \delta\eta(x - a) \in C - a$, i.e. $(1 + \delta\eta)(x - a) \in C - a$.

Thus, putting $\delta_1 = \delta\eta/2$ we obtain the requested inclusion. □

Lemma 16. *Let C be a convex set with $0 \in C$. Let z be an arbitrary point. Suppose that $y \in (1 - \delta)C$ with $0 < \delta < 1$ and $x = \lambda y + (1 - \lambda)z$ with $0 < \lambda \leq 1$. Then*

$$x \in (1 - \lambda\delta) \operatorname{co}(z, C).$$

Proof. This conclusion follows from the representation

$$\lambda y + (1 - \lambda)z = (1 - \lambda\delta) \left(\frac{\lambda(1 - \delta)}{1 - \lambda\delta} \frac{y}{1 - \delta} + \frac{1 - \lambda}{1 - \lambda\delta} z \right). \quad \square$$

Proof of (iii). Since C_1 and C_2 are bounded, there is a q , $0 < q < \infty$, such that $\alpha(C_1), \alpha(C_2) \leq q$.

We shall apply Theorem 13. Let $\varepsilon > 0$. Let $\{x_n\} \subset C^3$, $\operatorname{sep}(x_n) > \varepsilon$ and $x_n \rightarrow x$ weakly. Write: $x_n = \lambda_n y_n + (1 - \lambda_n)z_n$, where $\{y_n\} \subset C_1$, $\{z_n\} \subset C_2$, $0 \leq \lambda_n \leq 1$. By the weak compactness, we may pass to a subsequence such that $y_n \rightarrow y$ weakly and $z_n \rightarrow z$ weakly, $y \in C_1$, $z \in C_2$. We can also assume that $\lambda_n \rightarrow \lambda$. Thus, $x = \lambda y + (1 - \lambda)z$. Since C_1 and C_2 are bounded, without loss of generality we can assume that

$$x_n = \lambda y_n + (1 - \lambda)z_n.$$

Write $A = \{x_n\}$, $Y = \{y_n\}$, $Z = \{z_n\}$. Therefore,

$$A \subset \lambda Y + (1 - \lambda)Z.$$

By $\operatorname{sep}(x_n) > \varepsilon$ and the properties of the Kuratowski measure of noncompactness, we get

$$\varepsilon < \alpha(A) \leq \lambda \alpha(Y) + (1 - \lambda) \alpha(Z).$$

Thus we have either

$$(8) \quad \lambda \alpha(Y) > \frac{\varepsilon}{2},$$

or

$$(9) \quad (1 - \lambda) \alpha(Z) > \frac{\varepsilon}{2}.$$

Let (8) be satisfied. It implies

$$\alpha(Y) > \frac{\varepsilon}{2\lambda} > \frac{\varepsilon}{2q}.$$

By the first inequality, there is a subsequence of $\{y_n\}$ (denoted again by $\{y_n\}$) such that $\operatorname{sep}(y_n) > \varepsilon/4$. By (NUC) of C_1 , we can choose $\delta_1 = \delta_1(\varepsilon/4)$, $0 < \delta_1 < 1$ such that $y - a \in (1 - \delta_1)(C_1 - a)$. It follows from Lemma 16 that

$$x - a \in (1 - \lambda\delta_1)(C^3 - a),$$

hence

$$x - a \in \left(1 - \frac{\delta_1 \varepsilon}{2q}\right) (C^3 - a).$$

Let (9) be satisfied. Then

$$\alpha(Z) > \frac{\varepsilon}{2} \quad \text{and} \quad 1 - \lambda > \frac{\varepsilon}{2q}.$$

By (NUC) of C_2 , choose a $\delta_2 = \delta_2\left(\frac{\varepsilon}{4}\right)$. As before, we obtain that

$$x - b \in \left(1 - \frac{\delta_2 \varepsilon}{2q}\right) (C^3 - b).$$

According to Lemma 15, there is an η , $0 < \eta < 1$, dependent on C , a and b , such that

$$x - a \in \left(1 - \frac{\delta_2 \eta \varepsilon}{4q}\right) (C^3 - a).$$

Finally, put

$$\delta = \min \left(\frac{\delta_1 \varepsilon}{2q}, \frac{\delta_2 \eta \varepsilon}{4q} \right).$$

Then

$$x - a \in (1 - \delta)(C^3 - a).$$

Hence by Theorem 14, C^3 is (NUC) with respect to a . This, and Corollary 7, together imply (iii).

(iv) Let $\varepsilon > 0$. Suppose that $\{x_n\} \subset C^4$, $\text{sep}(x_n) > \varepsilon$, $x_n \rightarrow x$ weakly. Choose $0 < \delta < 1$ corresponding to ε in the definition of (NUC) for C_1 with respect to a . Since K is compact, we can assume as before that

$$x_n = \lambda y_n + (1 - \lambda)z,$$

where $y_n \rightarrow y$ weakly, $y \in C_1$, $z \in K$. Then

$$\lambda \text{sep}(y_n) > \varepsilon.$$

Hence

$$\text{sep}(y_n) > \varepsilon \quad \text{and} \quad \lambda > \frac{\varepsilon}{q}, \quad \text{where} \quad q = \text{diam } C_1.$$

By Lemma 16,

$$x - a \in \left(1 - \frac{\delta \varepsilon}{q}\right) (C - a).$$

Thus C^4 is (NUC). □

If C_1 is the closed unit ball B , $K = \{x_0, -x_0\}$, $x) - \notin B$ then (iv) is proved in [7]. A brief proof of this fact based also on the characterization of Huff [1] has been given independently by Montesinos and Torregrosa [5].

Remark 17. The fact that both C_1 and C_2 are (NUC) with respect to interior points is essential in (ii) and (iii) as the following example shows.

Let $X = l^2$ and let C_1 be the closed unit ball. Write:

$$C_2 = \left\{ x = (x_n) : x_1 = 1 \text{ and } \sum_{i=2}^{\infty} x_i^2 \leq 1 \right\}.$$

The set C_2 , is (NUC) with respect to the point $(1, 0, 0, 0, \dots)$. It follows from Corollary 8 that neither $C^2 = C_1 + C_2$ nor $C^3 = \text{co}(C_1, C_2)$ is (NUC) with respect to an interior point.

Corollary 18. *Let X be a Banach space which contains a bounded closed convex set C , which is (NUC) with respect to an interior point. Then X admits an equivalent norm which is (NUC).*

Proof. Without loss of generality we may assume that $0 \in \text{int } C$. Put $B = C \cap (-C)$. By Theorem 14 (i), B is (NUC) with the center 0. Clearly, B is (NUC) with respect to any equivalent norm in X . Since B is a bounded symmetric convex set with $0 \in \text{int } B$, the Minkowski functional of B defines an equivalent norm in X which is (NUC). $\square$

References

- [1] K. GOEBEL and T. SEKOWSKI, The modulus of non-compact convexity. Ann. Univ. M. Curie-Skłodowska, Sect. A, **38**, 41–48 (1984).
- [2] R. HUFF, Banach spaces which are nearly uniformly convex. Rocky Mount. J. Math. (4) **10**, 743–749 (1980).
- [3] D. N. KUTZAROVA and S. ROLEWICZ, On drop property for convex sets. Arch. Math. **56**, 501–511 (1991).
- [4] V. MONTESINOS, Drop property equals reflexivity. Stud. Math. **87**, 93–100 (1987).
- [5] V. MONTESINOS and J. R. TORREGROSA, A uniform geometric property in Banach spaces. Rocky Mountain J. Math., **22** (1992), 683–690.
- [6] S. ROLEWICZ, On drop property. Stud. Math. **85**, 27–35 (1987).
- [7] S. ROLEWICZ, On Δ -uniform convexity and drop property. Stud. Math. **87**, 181–191 (1987).