PROPERTY (β) IMPLIES NORMAL STRUCTURE OF THE DUAL SPACE

by

D. KUTZAROVA — E. MALUTA — S. PRUS

A.M.S. subject classification: 46B20.

We prove that property (β) of Rolewicz implies normal structure of the dual space and we characterize spaces which are duals of spaces with property (β) .

1. Introduction.

The notion of normal structure has been introduced by Brodskii and Milman [1] in connection with fixed point theory. For a bounded subset A of a normed space X , the Chebyshev self-radius of A is

$$r(A) = \inf \{ \sup \{ \|x - y\| : y \in A \} : x \in \operatorname{conv} A \}.$$

A normed space X is said to have normal structure if $r(A) < \operatorname{diam}(A)$ for every non-singleton bounded subset A of X . If moreover $r(A)/\operatorname{diam}(A)$ is bounded away from 1, X is said to have uniformly normal structure. In [14] it was proved that, for a Banach space X , uniformly normal structure implies (normal structure and) reflexivity, and it was introduced a coefficient $D(X)$ of sequentially uniform normal structure defined by

$$D(X) = \sup \left\{ \limsup_i d(x_{i+1}, \operatorname{conv}\{x_n\}_1^i) : \{x_n\} \subseteq X, \operatorname{diam}\{x_n\} = 1 \right\}$$

The property $D(X) < 1$ is weaker than uniformly normal structure; nevertheless it too implies (normal structure and) reflexivity, while the converse is not true.

It is known that spaces with a certain degree of uniform rotundity have normal structure, but in this paper we are mainly interested in properties of uniform rotundity of the space X which assure normal structure of its dual space X^* .

In fact normal structure is not a self-dual property.

Precisely in 1972 Bynum [2] proved that the space $X = (l_2, \|\cdot\|_{2,1})$, where

$$\|x\|_{2,1} = \|x^+\|_2 + \|x^-\|_2$$

($\|\cdot\|_2$ is the l_2 -norm, x^+ and x^- the positive and negative parts of x) has normal structure, while its dual lacks it.

A classical result by Turett [21] states that if the characteristic of convexity of X , $\varepsilon_0(X)$, is less than 1, then both X and X^* are superreflexive spaces with normal structure (even uniformly normal structure [7], [17]).

In particular uniformly rotund (UR) spaces and their (uniformly smooth) duals have uniformly normal structure.

Also the property of k -uniform rotundity (k -UR) of Sullivan [20] implies superreflexivity and uniformly normal structure of the space, but recently Smith and Turett [19] proved that Bynum's space is 2-UR (hence k -UR for any $k > 2$). Therefore even quite strong properties of uniform rotundity of X fail to assure normal structure of the dual.

Aim of this paper is to prove that property (β) of Rolewicz [18], that is a condition of uniform rotundity of X weaker than UR and which is not isometrically comparable with k -UR (while it is isomorphically weaker than both, i.e. it does not imply superreflexivity), does imply normal structure of X^* .

Among properties of uniform rotundity weaker than property (β) , one of the most extensively examined in the literature is the so called nearly uniform convexity (NUC) [6], which is known to imply $D(X) < 1$ [3]. We remark that Bynum's space is a NUC space whose dual lacks normal structure.

2. Definitions and preliminaries.

The following properties and some of their generalizations will be used in the paper.

The norm of a space X is called uniformly Kadec-Klee (UKK) if for every $\varepsilon > 0$ there exists $\delta < 1$ such that if $\{x_n\}$ is a sequence in the unit ball B_X of X with $\text{sep}(\{x_n\}) = \inf\{\|x_i - x_j\| : i \neq j\} > \varepsilon$ and $x_n \xrightarrow{w} x$, then $\|x\| < \delta$.

A nearly uniformly convex (NUC) space is a reflexive space whose norm is UKK (see [6]).

For our approach it is more convenient to use, instead of the original definition, a condition involving Kuratowsky's measure of non-compactness.

We recall that Kuratowsky's measure of non-compactness of a set A in a Banach space X , $\alpha(A)$, is the infimum of those $\varepsilon > 0$ for which A admits a finite covering by sets whose diameter is less than ε .

A space X is called (following [5]) Δ -uniformly convex (or non-compactly uniformly convex) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each convex subset E of the closed unit ball B_X of X such that $\alpha(E) > \varepsilon$ we have $\inf\{\|x\| : x \in E\} < 1 - \delta$.

It is easy to see that Δ -uniform convexity coincides with Huff's NUC.

Dual spaces of *NUC* spaces have been characterized by Prus [16] by the following property, that has been called nearly uniform smoothness.

A space X is called nearly uniformly smooth (*NUS*) if for every $\varepsilon > 0$ there exists $\eta > 0$ such that for each $t \in [0, \eta]$ and each basic sequence $\{x_n\}$ in B_X there is $k > 1$ such that

$$\|x_1 + tx_k\| \leq 1 + \varepsilon t.$$

In [18], studying the relationship between *NUC* and drop property, Rolewicz defined property (β) . In a Banach space X with closed unit ball B_X , the drop $D(x, B_X)$ defined by an element $x \in X \setminus B_X$ is the set $\text{conv}(\{x\} \cup B_X)$, and we set

$$R(x, B_X) = D(x, B_X) \setminus B_X.$$

X is said to have property (β) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $1 < \|x\| < 1 + \delta$ implies $\alpha(R(x, B_X)) < \varepsilon$.

Rolewicz [18] has shown that *UR* implies property (β) which in turn implies *NUC*. Moreover, (β) is isomorphically different from both of them (see [8], [15] and [9]). In [11] an isomorphic characterization of (β) is given in the spirit of [16], in terms of (p, q) -estimates. More precisely it has been proved there that if the norm in a Banach space is both *NUS* and *NUC*, then it possesses property (β) . The converse is not isometrically true, but (β) (even (β, ε) for some fixed ε , $0 < \varepsilon < 1$) implies (∞, q) -estimates [11], which for a separable Banach space implies existence of an equivalent *NUS* norm [16].

We shall use in the sequel the following characterization of property (β) .

PROPOSITION 1. ([10]) *A Banach space X has property (β) if and only if for each $\varepsilon > 0$ there exists δ , $0 < \delta < 1$, such that for every element $x \in B_X$ and every sequence $\{x_n\} \subset B_X$ with $\text{sep}(\{x_n\}) > \varepsilon$, there is an index i such that $\|x + x_i\|/2 \leq 1 - \delta$.*

3. Results.

We start giving a characterization of spaces duals of spaces with property (β) .

DEFINITION 1. *A Banach space X is said to have property (β^*) if for every $\varepsilon > 0$ there exists $\eta > 0$ such that if $0 < t < \eta$, $\{x_n\}$ is a sequence in B_X , $\{x_n\}$ weakly convergent to x , and $\{y_n\}$ is a basic sequence in B_X , then for some $k \geq 1$*

$$\|x\| + \|x_k + ty_k\| \leq 2 + \varepsilon t.$$

Remark 1. Taking all the x'_n s equal to x , with $\|x\| = 1$, it is easy to see that (β^*) implies *NUS*, which in turn implies reflexivity.

THEOREM 1. *A Banach space X has property (β^*) if and only if its dual space X^* has property (β) .*

Proof. Let us assume first that X has property (β^*) . For a given $\varepsilon > 0$ we obtain some $\eta > 0$ which corresponds to $\varepsilon/12$. Let us fix $0 < t < \eta$, an element $x^* \in B_{X^*}$ and a sequence $\{x_n^*\}$ in B_{X^*} with $\text{sep}(\{x_n^*\}) > \varepsilon$.

Since X is reflexive, we can make the following assumptions:

- 1) $\{x_n^*\} \xrightarrow{w} z^*$;
- 2) $\|x_n^* - z^*\| \geq \varepsilon/2$ for all n ;
- 3) there are elements $z_n \in B_X$ such that $(x_n^* - z^*)(z_n) \geq \varepsilon/2$, $z_n \xrightarrow{w} z$, and $\{z_n - z\}$ is a basic sequence.

Now we take $x_n \in B_X$ with $(x^* + x_n^*)(x_n) = \|x^* + x_n^*\|$ and assume that $x_n \xrightarrow{w} x$. Thus, for k large enough, $|(x_k^* - z^*)(z)| \leq \varepsilon/12$, $|z^*(z_k - z)| \leq \varepsilon/12$ and $|x^*(x_k - x)| \leq \varepsilon t/12$. Therefore from Definition 1 we get some k for which

$$\begin{aligned} \|x^* + x_k^*\| &= x^*(x_k) + x_k^*(x_k) = x^*(x) + x^*(x_k - x) + x_k^*(x_k) \leq \\ &\leq \|x\| + \varepsilon t/12 + x_k^*(x_k + t(z_k - z)) - tx_k^*(z_k - z) \leq \\ &\leq \|x\| + \|x_k + t(z_k - z)\| + \varepsilon t/12 - \varepsilon t/3 \leq 2 - \varepsilon t/6. \end{aligned}$$

By Proposition 1, this shows that X^* has property (β) .

Let us now assume that X^* has property (β) . For $0 < \varepsilon$, $\gamma < 1$, Proposition 1 gives us $\delta > 0$ corresponding to $(1 - \gamma)\varepsilon$. We fix t , $0 < t < \delta/(1 - \varepsilon)$ and consider sequences $\{x_n\}$ and $\{y_n\}$ in B_X such that $x_n \xrightarrow{w} x$ and $\{y\}$ is a basic sequence. There exists $x_n^* \in B_{X^*}$ with $x_n^*(x_n + ty_n) = \|x_n + ty_n\|$ and $x^* \in B_{X^*}$ for which $x^*(x) = \|x\|$. We may assume that $x_n^* \xrightarrow{w} z^*$, $|z^*(y_n)| \leq \gamma\varepsilon/2$ and $|x^*(x_n - x)| \leq \delta$.

Let us consider the following cases:

1. $\|x_k^* - z^*\| \leq (1 - \gamma/2)\varepsilon$ for some k . Then

$$\begin{aligned} \|x\| + \|x_k + ty_k\| &\leq 1 + x_k^*(x_k) + tx_k^*(y_k) \leq \\ &\leq 2 + t((x_k^* - z^*)(y_k) + \gamma\varepsilon/2) \leq 2 + \varepsilon t. \end{aligned}$$

2. $\|x_k^* - z^*\| > (1 - \gamma/2)\varepsilon$ for all k . Then we may assume that $\text{sep}(\{x_n^*\}) > (1 - \gamma)\varepsilon$. Therefore by Proposition 1 there is a k such that

$$\begin{aligned} \|x\| + \|x_k + ty_k\| &\leq x^*(x_k) + \delta + x_k^*(x_k) + tx_k^*(y_k) \leq \\ &\leq \|x^* + x_k^*\| + \delta + t \leq 2(1 - \delta) + \delta + t \leq 2 + \varepsilon t. \quad \square \end{aligned}$$

The following theorem is our main result.

THEOREM 2. *If a Banach space X has property (β) then, for its dual space X^* , $D(X^*) < 1$.*

We prove a slightly stronger result. Namely, in the spirit of Proposition 1, we shall say that X has property (β, γ) for some $\gamma > 0$ if there exists a $\delta > 0$ such that for every $x \in B_X$ and every $\{x_n\} \subset B_X$ with $\text{sep}(\{x_n\}) > \gamma$ there exists an i such that $\|x + x_i\|/2 < 1 - \delta$. We have that (β, γ) implies reflexivity. In fact, it is easy to prove that it implies property (NUC, γ) of [12], which is known to assure reflexivity.

Moreover the second part of the proof of Theorem 1 shows that if X has (β, γ) for some $\gamma \in (0, 1)$ then X^* has the weak version of (β^*) which is obtained replacing, in Definition 1, “for every $\varepsilon > 0$ ” by “there exists an $\varepsilon \in (0, 1)$ ”. We call this property (β^*, ε) .

THEOREM 2 bis. *If a Banach space X has property (β, γ) for some γ , $0 < \gamma < 1$, then $D(X^*) < 1$.*

Proof. By reflexivity we may exchange X and X^* . Assume $D(X) = 1$. Since in [14] it was proved that $D(X)$ can be equivalently defined as

$$D(X) = \sup \left\{ \liminf_i d(x_{i+1}, \text{conv}\{x_n\}_1^i) : \{x_n\} \subseteq X, \text{diam}\{x_n\} = 1 \right\}$$

as in [3] it is easy to see that, for each $\delta \in (0, 1)$, we may choose a sequence $\{x_n\}$ such that

- 1) $\text{diam}(\{x_n\}) < 1$;
- 2) $x_n \xrightarrow{w} 0$;
- 3) $\|x_n\| \geq 1 - \delta$ for every n ;
- 4) $\{x_n\}$ is a basic sequence.

For each n take an $x_n^* \in X^*$ with $\|x_n^*\| = 1$ and

$$x_n^*(x_n) = \|x_n\|.$$

By reflexivity, we may assume, passing to a subsequence if necessary, that $\{x_n^*\}$ converges weakly to some x^* . According to 2), there exists an n_1 such that for every $n \geq n_1$

$$|x^*(x_n)| < \delta$$

and let $n_2 > n_1$ such that for every $n \geq n_2$

$$|(x_n^* - x^*)(x_{n_1})| < \delta.$$

Therefore we have for every $n \geq n_2$

$$|x_n^*(x_{n_1})| < 2\delta.$$

Set $\{x_{n_k}\} = \{x_{n_1}, x_{n_2}, x_{n_2+i} : i > 0\}$. Since we are assuming that X^* , has property (β, γ) (remember that we exchanged X and X^* by sake of notational simplicity), there is an $\varepsilon \in (0, 1)$ such that for some $t > 0$ and $k > 1$ we have

$$\begin{aligned} 2 + \varepsilon t &\geq \|x_{n_1}\| + \|x_{n_k} - x_{n_1} + tx_{n_k}\| \geq \\ &\geq 1 - \delta + x_{n_k}^*(x_{n_k} - x_{n_1} + tx_{n_k}) \geq 2 + t - \delta(4 + t), \end{aligned}$$

a contradiction because δ is arbitrary and $\varepsilon < 1$. $\square$

Remark 2. As we mentioned above, (β) implies neither NUS nor superreflexivity, hence Theorem 2 is not a consequence of results in [3] or in [21].

Remark 3. In [10] the notion of k - β space, $k \geq 1$, and the notion of k - NUC space, $k \geq 2$, have been denned, where 1- β coincides with property (β) . k - β implies $(k+1)$ - NUC which in turn implies $(k+1)$ - β . Moreover k - UR implies k - β . Hence Bynum's space is a k - β space for every $k \geq 2$ though its dual lacks normal structure.

Remark 4. Any of the properties k - β or k - NUC imply NUC . Therefore the dual space of Bynum's is a NUC space which lacks normal structure.

We recall that, given an integer $k \geq 2$, a Banach space X is called k -nearly uniformly convex (k - NUC) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence $\{x_n\} \subseteq B_X$ with $\text{sep}(\{x_n\}) > \varepsilon$, there exist indices $\{n_i\}_{i=1}^k$ and scalars $\gamma_i \geq 0$, $i = 1, \dots, k$ with $\sum_1^k \gamma_i = 1$ such that $\left\| \sum_1^k \gamma_i x_{n_i} \right\| \leq 1 - \delta$.

Remark 5. A space can be 2- UR and fail to be 2- NUC (see [10] for an example). Bynum's space turns out to be also 2- NUC .

THEOREM 3. *The space $X = (l_2, \|\cdot\|_{2,i})$ is a 2- NUC space.*

Proof. For each $\varepsilon > 0$ let $\delta_2(\varepsilon)$ be the $(l_2, \|\cdot\|_2)$ modulus of convexity, i.e.

$$\delta_2(\varepsilon) = \inf\{1 - \|x + y\|_2/2 : \|x\|_2 \leq 1; \|y\|_2 \leq 1; \|x - y\|_2 > \varepsilon\}$$

and set

$$\delta = \delta_2(\varepsilon/4).$$

Let $\{x_n\} \subset B_X$ with $\text{sep}(\{x_n\}) > \varepsilon$. Passing to a subsequence if necessary, we may assume that the separation of $\{x_n^+\}$ or $\{x_n^-\}$ is bigger than $\varepsilon/4$; say $\text{sep}(\{x_n^+\}) > \varepsilon/4$. Moreover we may assume that $\|x_n^+\| \rightarrow a$ and $\|x_n^-\| \rightarrow b$. Obviously $a + b \leq 1$ and $a > \varepsilon/8$.

If $a = 1$ the proof is trivial. Otherwise choose $\eta > 0$ such that $a + \eta \leq 1$ and $\eta(1 - \delta) < (a - \varepsilon/8)\delta$. For i and j large enough, $\|x_i^+\|_2 \leq a + \eta$ and $\|x_j^+\|_2 \leq a + \eta$ imply

$$\|x_i^+ + x_j^+\|_2/2 \leq (a + \eta)(1 - \delta_2(\varepsilon/(4(a + \eta)))) \leq (a + \eta)(1 - \delta).$$

Hence, taking into account that $\|x_n^-\|_2 \rightarrow b$,

$$\|x_i + x_j\|_{2,1}/2 = \|x_i^+ + x_j^+\|_2/2 + \|x_i^- + x_j^-\|_2/2 \leq 1 - \varepsilon\delta/16. \quad \square$$

Now we try to complete the scheme of implications between *NUC* type properties and normal structure.

Actually the two strongest results in this line are related one to the characteristic of non-compact convexity of the space and the other one to the weakly uniform Kadec-Klee property (*WUKK*) of the norm of the space.

We recall that the modulus of non-compact convexity of X , $\Delta_X(\varepsilon)$, is defined ([5]) for every $\varepsilon \in [0, 2]$ by

$$\Delta_X(\varepsilon) = \inf\{1 - \inf\{\|x\| : x \in A\} : A \subseteq B_X, A \text{ convex}, \alpha(A) \geq \varepsilon\}$$

and the related characteristic of non-compact convexity is

$$\varepsilon_1(X) = \sup\{\varepsilon : \Delta_X(\varepsilon) = 0\}.$$

It is known that $\varepsilon_1(X) < 1$ implies reflexivity and normal structure [5].

As about *WUKK* ([4]), it is the property obtained from the definition of *UKK* by replacing “for every $\varepsilon \in (0, 1)$ ” with “there exists $\varepsilon \in (0, 1)$ ” i.e. the norm of X is *WUKK* if there exist $\varepsilon, \delta \in (0, 1)$ such that if $\{x_n\}$ is a sequence in B_X , $x_n \xrightarrow{w} x$ and $\text{sep}(\{x_n\}) > \varepsilon$, then $\|x\| \leq 1 - \delta$.

WUKK is known to imply normal structure for the weakly compact convex sets of X .

Here we consider the property *WUKK'* obtained replacing, in the above definition of *WUKK*, “ $\text{sep}(\{x_n\}) > \varepsilon$ ” with “ $\liminf_n \|x_n - x\| > \varepsilon$ ”.

DEFINITION 2. *Let X be a Banach space. Its norm is said to have property *WUKK'* if there exist $\varepsilon, \delta \in (0, 1)$ such that for every sequence $\{x_n\} \subset B_X$, $x_n \xrightarrow{w} x$, with $\liminf_n \|x_n - x\| > \varepsilon$ we have $\|x\| \leq 1 - \delta$.*

PROPOSITION 2. *Let X be a Banach space. Property $\varepsilon_1(X) < 1$ implies *WUKK* which in turn implies *WUKK'*.*

Proof. To prove the first implication take ε such that $\varepsilon_1(X) < \varepsilon < 1$. If $\{x_n\} \subset B_X$, $x_n \xrightarrow{w} x$ and $\text{sep}(\{x_n\}) > \varepsilon$, then $\alpha(\text{cl conv}(\{x_n\}_k^{+\infty})) > \varepsilon$, hence, if $\delta > \Delta_X(\varepsilon)$,

$$\inf\{\|x\| : x \in \text{cl conv}(\{x_n\}_k^{+\infty})\} \leq 1 - \delta \quad \text{for every } k.$$

Since X is reflexive,

$$\{x\} = \bigcap_k cl \operatorname{conv}(\{x_n\}_k^{+\infty}), \quad \text{hence} \quad \|x\| < 1 - \delta.$$

The second claim is proved if we show that if $\{x_n\}$ is a sequence in B_X , $x_n \xrightarrow{w} x$, such that $\liminf_n \|x_n - x\| > \varepsilon$, then for any $\gamma \in (0, \varepsilon)$ there exists a subsequence $\{x_{n_k}\}$ with $\operatorname{sep}(\{x_{n_k}\}) > \gamma$.

Fix any $\gamma \in (0, \varepsilon)$. We may assume that $x_n \xrightarrow{w} 0$ and that $\|x_n\| > (\varepsilon + \gamma)/2$ for every n .

Let, for any n , $x_n^* \in X^*$ be such that $\|x_n^*\| = 1$ and $x_n^*(x_n) = \|x_n\|$.

Set $x_{n_1} = x_1$ and choose n_2 such that $|x_{n_1}^*(x_{n_2})| < (\varepsilon - \gamma)/2$ for every $n > n_2$.

By induction, construct a sequence such that $|x_{n_k}^*(x_{n_h})| < (\varepsilon - \gamma)/2$ for each $k < h$.

Then, assuming $k < h$,

$$\|x_{n_k} - x_{n_h}\| \geq |x_{n_k}^*(x_{n_k} - x_{n_h})| > \gamma$$

i.e. $\operatorname{sep}(\{x_{n_k}\}) > \gamma$ as requested. $\square$

THEOREM 4. *Let X be a reflexive Banach space. If its norm satisfies property $WUKK'$, then $D(X) < 1$.*

Proof. As in Theorem 2, assuming $D(X) = 1$, it is easy to see that for any $\eta < 1$ we can find a sequence $\{x_n\}$ such that

- 1) $\operatorname{diam}(\{x_n\}) < 1$;
- 2) $x_n \xrightarrow{w} 0$;
- 3) $\|x_n\| > \eta$ for every n .

Now let ε, δ be determined by the property $WUKK'$ of X and set $\eta = \max(\varepsilon, 1 - \delta)$. For the $\{x_n\}$ that satisfies 1), 2) and 3) with respect to η , set $z_n = x_1 - x_n$. $\{z_n\} \subset B_X$ because of 1), $z_n \xrightarrow{w} x_1$ and $\liminf_n \|z_n - x_1\| = \liminf_n \|x_n\| > \varepsilon$. Therefore $\|x_1\| < 1 - \delta \leq \eta$, which contradicts 3). $\square$

COROLLARY 5. *If a Banach space X has $\varepsilon_1(X) < 1$, then $D(X) < 1$.*

Remark 6. Property (β) (hence the weaker properties NUC and $\varepsilon_1(X) < 1$) doesn't imply uniformly normal structure. In fact Day's example of a reflexive non superreflexive Banach space has been proved to have property (β) ([8]) while, having an unconditional basis, it cannot have uniformly normal structure because it is not superreflexive ([3]).

We have already mentioned in Remark 2 that (β^*) does not imply NUC . The following example shows that in general neither $WUKK'$, that is the weakest

property of type UKK we considered, is implied by (β^*) . Significance is added to the example by the result that, in spaces which possess the weak Opial property, (β^*) does imply $WUKK'$.

EXAMPLE. There exists a Banach space X which has property (β) but the norm of its dual space is not $WUKK'$.

Let us start from the following norm in $\mathbb{R}^2$:

$$\|(\alpha_1, \alpha_2)\|_0 = \left(\frac{21}{22}(\alpha_1 + \alpha_2)^2 + \frac{1}{22}(\alpha_1 - \alpha_2)^2 \right)^{1/2}.$$

By X we denote the space l^2 with the equivalent norm

$$\|x\| = \sup_{k \geq 1} (\|(\alpha_1, \alpha_k)\|_0^2 + c\|Q_k x\|_2^2)^{1/2},$$

where $x = \{\alpha_n\}$, $Q_k x = \{\alpha_n\}_{n \geq k}$, $c = (20\sqrt{79} - 79)/121$ and $\|\cdot\|_2$ is the l_2 -norm.

In order to prove that X has property (β) , we prove first the following

Claim. For every $\varepsilon > 0$ there exists a positive $\delta < 1$ such that if $u, v, w \in X$ are blocks in X (i.e. they have finite supports), with $\max\{\max\{\text{supp } u\}, \max\{\text{supp } v\}\} + 1 < \min\{\text{supp } w\}$, $\|u\| \leq 1$, $\|v+w\| \leq 1$ and $\|w\|_2 \geq \varepsilon/2$, then $\|u+v+w\| \leq 2(1-\delta)$.

To obtain an upper estimate for $\|u+v+w\|$ it suffices to consider the following two expressions, where we set $u = \{\alpha_n\}$, $v = \{\beta_n\}$ and $w = \{\gamma_n\}$.

$$\begin{aligned} \text{Case 1.} \quad & \|(\alpha_1 + \beta_1, \alpha_k + \beta_k)\|_0^2 + c\|Q_k(u+v+w)\|_2^2 \leq \\ & \leq (\|(\alpha_1, \alpha_k)\|_0 + \|(\beta_1, \beta_k)\|_0)^2 + c(\|Q_k u\|_2 + \|Q_k v\|_2)^2 + c\|w\|_2^2 \leq \\ & \leq (1 + (\|(\beta_1, \beta_k)\|_0^2 + c\|Q_k v\|_2^2)^{1/2})^2 + c\|w\|_2^2 \leq \\ & \leq (1 + (1 - c\|w\|_2^2)^{1/2})^2 + c\|w\|_2^2 \leq 2(1 + (1 - c\varepsilon^2/4)^{1/2}). \end{aligned}$$

$$\text{Case 2.} \quad \|(\alpha_1 + \beta_1, \gamma_k)\|_0^2 + c\|Q_k w\|_2^2.$$

We have $|\alpha_1| \leq 1$ and $\|(\beta_1, \gamma_k)\|_0 \leq 1$. Moreover $|\beta_1| \leq (1 - c\varepsilon^2/4)$ because $\beta_1^2 + c\|w\|_2^2 \leq 1$.

The two dimensional subspaces spanned by the first and k -th basic vectors are isometric and strictly convex for $k > 2$. It follows that there is a constant $\delta_0(\varepsilon)$, $0 < \delta_0(\varepsilon) < 1$, such that $\|(\alpha_1 + \beta_1, \gamma_k)\|_0 \leq 2(1 - \delta_0(\varepsilon))$.

If $c\|Q_k w\|_2^2 \leq \delta_0(\varepsilon)$, then

$$\|(\alpha_1 + \beta_1, \gamma_k)\|_0^2 + c\|Q_k w\|_2^2 \leq 4 - \delta_0(\varepsilon)(7 - 4\delta_0(\varepsilon));$$

if $c\|Q_k w\|_2^2 > \delta_0(\varepsilon)$, then

$$\begin{aligned} & \|(\alpha_1 + \beta_1, \gamma_k)\|_0^2 + c\|Q_k w\|_2^2 \leq (1 + \|(\beta_1, \gamma_k)\|_0)^2 + c\|Q_k w\|_2^2 \leq \\ & \leq (1 + (1 - c\|Q_k w\|_2^2)^{1/2})^2 + c\|Q_k w\|_2^2 \leq 2(1 + (1 - \delta_0(\varepsilon))^{1/2}). \end{aligned}$$

Therefore the claim is proved.

Now we take $x \in B_X$ and $\{x_n\} \subset B_X$ with $\text{sep}(\{x_n\}) \geq \varepsilon$. Since X is reflexive, there exists a subsequence, that we still call $\{x_n\}$ by sake of simplicity, which converges weakly to some y .

The next step is an application of the gliding hump technique described for example in [13], pp. 6–7. However in our case we do not need an infinite sequence of blocks. Namely, given ε and δ as in the claim, and having γ with $0 < \gamma < \min\{3/2^{3/2} - 1, \varepsilon/4, \delta/(4 - 2\delta)\}$, we choose blocks u, v such that $\|x - u\| \leq \gamma$, $\|y - v\| \leq \gamma/2$.

Since $\{x_n - y\} \xrightarrow{w} 0$, there is n_1 such that $\|x_{n_1} - y - w_1\| < \gamma/2$ for some block w_1 with

$$\min\{\text{supp } w_1\} > \max\{\max\{\text{supp } u\}, \max\{\text{supp } v\}\} + 1.$$

Similarly we obtain $n_2 > n_1$ and a block w_2 such that $\|x_{n_2} - y - w_2\| \leq \gamma/2$ and $\min\{\text{supp } w_2\} > \max\{\text{supp } w_1\}$. Then $\|u\| \leq 1 + \gamma$, $\|v + w_i\| < 1 + \gamma$, $i = 1, 2$, and $\|w_1 - w_2\| \geq \varepsilon - \gamma > 3\varepsilon/4$.

But the first coordinate of $w_1 - w_2$ equals 0, and $\text{supp } w_1, \text{supp } w_2$ are disjoint. Therefore $3\varepsilon/4 < \|w_1 - w_2\|_2 = (\|w_1\|_2^2 + \|w_2\|_2^2)^{1/2}$.

It follows that $\|w_i\|_2 > 3\varepsilon/2^{5/2}$ for some i . Hence $\|w_i/(1 + \gamma)\|_2 > \varepsilon/2$ and in consequence $\|u + v + w_i\| \leq 2(1 + \gamma)(1 - \delta)$.

Therefore $\|(x + x_{n_i})/2\| \leq (1 + \gamma)(1 - \delta) + \gamma < 1 - \delta/2$, which in light of Proposition 1 shows that X has property (β) .

Let us now consider the functionals

$$x_1^*(x) = \alpha_1, \quad x_k^*(x) = \frac{10}{11}\alpha_k, \quad k > 1,$$

where $x = \{\alpha_n\}$. We have $\|x_1^* + x_k^*\| \leq 1$ for all $k > 1$. Moreover $x_1^*((1, 0, \dots)) = 1$ and $x_k^*((\sqrt{79}/10 - 1, 0, \dots, 0, 11/10, 0, \dots)) = 1$, which shows that $\|x_1^*\| \geq 1$, $\|x_k^*\| \geq$

1 for all k . Since $x_n^* \xrightarrow{w} 0$, it follows that X^* is not $WUKK'$.

We recall that a space X has the weak Opial property if

$$\liminf_n \|x_n\| \leq \liminf_n \|x + x_n\|$$

for every element x and every sequence $\{x_n\}$ with $x_n \xrightarrow{w} 0$ (this is the case if X has a bimonotone basis, i.e. a basis $\{e_n\}$ such that

$$\left\| \sum_{i=k}^m a_i e_i \right\| \leq \left\| \sum_{i=1}^n a_i e_i \right\|$$

for $1 \leq k \leq m \leq n$ and any scalars a_i , $i = 1, \dots, n$).

THEOREM 6. *If X is a Banach space with the weak Opial property, then if X has property (β^*) its norm is $WUKK'$.*

Proof. By property (β^*) we obtain $\eta > 0$ corresponding to $1/8$. For a fixed $0 < t < \eta$ we set $\gamma = (1 + t/3)/(1 + t/2)$.

Let now $\{x_n\}$ be a sequence in B_X with $x_n \xrightarrow{w} x$ and $\liminf_n \|x_n - x\| > \gamma$. Passing to a subsequence, we may assume that $\{x_n - x\}$ is a basic sequence. Clearly $\|x_n - x\| \leq 2$ for all n . Since X has property (β^*) , we can choose a subsequence $\{x_{n_k}\}$ such that

$$\|x\| + \|x_{n_k} + t(x_{n_k} - x)/2\| \leq 2 + t/8.$$

By the weak Opial property this implies the inequality

$$2 + t/8 \geq \|x\| + (1 + t/2) \liminf_k \|x_{n_k} - x\| \geq \|x\| + 1 + t/3.$$

Hence $\|x\| \leq 1 - 5t/24$, which shows that X is $WUKK'$. □

References

- [1] BRODSKII M. S., MILMAN D. R., *On the center of a convex set*, Dokl. Acad. Nauk. SSSR **59** (1948), 837–840.
- [2] BYNUM W. L., *A class of spaces lacking normal structure*, Comp. Math. **25** (1972), 233–236.
- [3] CASINI E., MALUTA E., *Normal type structures and superreflexivity*, Boll. U. M. I. **4-A 6** (1985), 303–308.
- [4] VAN DULST D., SIMS B., *Fixed points of nonexpansive mappings and Chebyshev centers in Banach spaces with norms of type KK* , in “Banach space theory and its applications”, Springer-Verlag, Berlin, 1983.
- [5] GOEBEL K., SEKOWSKI T., *The modulus of non-compact convexity*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A, **38** (1984), 41–48.
- [6] HUFF R., *Banach spaces which are nearly uniformly convex*. Rocky Mountain J. Math. **10** (1980), 743–749.
- [7] KHAMSI M. A., *Etude de la propriété du point fixe dans les espaces de Banach et les espaces métriques*, These de Doctoral de l’Université Paris VI, 1987.
- [8] KUTZAROVA D. N., *On condition (β) and Δ -uniform convexity*, C. R. Acad. Bulgar. Sci. **42** (1) (1989), 15–18.
- [9] KUTZAROVA D. N., *A nearly uniformly convex space which is not a (β) space*, to appear in Acta Univ. Carolinae, Math. et Phys. **30**, (1989), 95–98.
- [10] KUTZAROVA D. N., *k -(β) and k -nearly uniform convex Banach spaces*, J. Math. Anal. Appl. **162** (1991), 322–338.

- [11] KUTZAROVA D. N., *An isomorphic characterization of the property (β) of Rolewicz*, Note di Mat. **10** (1990), 2, 347–354.
- [12] LANDES T., *Normal structure and hereditariness properties. Functional Analysis and Approximation*, Edited by P. L. Papini, Bagni di Lucca (Italy) 16–20 May, 1988, 196–218.
- [13] Udenstrauss J., Tzafriri L., *Classical Banach spaces I*, Springer Verlag, New York, 1977.
- [14] MALUTA E., *Uniformly normal structure and related coefficients*, Pacific J. Math. **111** (1984), 357–369.
- [15] MONTESINOS V., TORREGROSA J. R., *A uniform geometric property in Banach spaces*, Rocky Mountain J. Math., **22** (1992), 683–690.
- [16] PRUS S., *Nearly uniformly smooth Banach spaces*, Boll. U. M. I. 3-B **7** (1989), 507–521.
- [17] PRUS S., *A remark on a theorem of Turett*, Bull. Polish Acad. Sci. Math. **36** (1988), 225–227.
- [18] ROLEWICZ S., *On Δ -uniform convexity and drop property*, Studia Math. **87** (1987), 181–191.
- [19] SMITH M., TURETT B., *Some examples concerning normal and uniform normal structure in Banach spaces*, J. Austral. Math. Soc. (ser A) **48** (1990), 223–234.
- [20] SULLIVAN E., *A generalization of uniformly rotund Banach spaces*, Canad. J. Math. **31** (1979), 628–636.
- [21] TURETT B., *A dual view of a theorem of Baillon*, Lect. Notes in Pure and Appl. Math. **80** (1982), 279–286.