

# ON STRONGLY ASYMPTOTIC $\ell_p$ SPACES AND MINIMALITY

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ABSTRACT. Let  $1 \leq p \leq \infty$  and let  $X$  be a Banach space with a strongly asymptotic  $\ell_p$  basis  $(e_i)$ . If  $X$  is minimal and  $1 \leq p < 2$ , then  $X$  is isomorphic to a subspace of  $\ell_p$ . If  $X$  is minimal and  $2 \leq p < \infty$ , or if  $X$  is complementably minimal and  $1 \leq p \leq \infty$ , then  $(e_i)$  is equivalent to the unit vector basis of  $\ell_p$  (or  $c_0$  if  $p = \infty$ ).

## 1. INTRODUCTION

The notion of minimality was introduced by H. Rosenthal. An infinite-dimensional Banach space  $X$  is *minimal* if every infinite-dimensional subspace has a further subspace isomorphic to  $X$ .

Let  $1 \leq p \leq \infty$ . A Banach space  $X$  with a basis  $(e_i)$  is *asymptotic  $\ell_p$*  [MT] if there exist  $C < \infty$  and an increasing function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that, for all  $n \in \mathbb{N}$ , every normalized block basis  $(x_i)_{i=1}^n$  of  $(e_i)_{i=f(n)}^\infty$  is  $C$ -equivalent to the unit vector basis of  $\ell_p^n$ . In this case  $(e_i)$  is called an *asymptotic  $\ell_p$  basis* for  $X$ .

The only known examples of minimal spaces were  $\ell_p$  ( $1 \leq p < \infty$ ) and  $c_0$  and their subspaces until the original Tsirelson space  $T^*$  [CJT], which is asymptotic  $\ell_\infty$ , was shown to be minimal [CJT]. Tsirelson's space  $T$  is asymptotic  $\ell_1$  and does not contain a minimal subspace [CO].

The next minimal space was constructed by Schlumprecht [S], and in [CKKM] a superreflexive version of  $S$  was given. Both versions are actually *complementably minimal*, i.e. every infinite-dimensional subspace has a complemented subspace isomorphic to the whole space. Gowers [G] included minimality in his partial classification of Banach spaces, which motivated a series of results relating minimality to subsymmetry [P], or to the number

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of nonisomorphic (resp. incomparable) subspaces of a Banach space [FR2], [F] (resp. [R2]).

We shall call a Banach space  $X$  with a basis  $(e_i)$  *strongly asymptotic*  $\ell_p$  if there exist  $C < \infty$  and an increasing function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that, for all  $n \in \mathbb{N}$ , every normalized sequence  $(x_i)_{i=1}^n$  of disjointly supported vectors from  $[(e_i)_{i=f(n)}^\infty]$  is  $C$ -equivalent to the unit vector basis of  $\ell_p^n$ .

In [CO] it was proved that the standard bases of  $T$  and its strongly asymptotic version (sometimes called modified Tsirelson's space) are equivalent. A new class of strongly asymptotic  $\ell_p$  spaces was introduced in [ADKM]. Sufficient conditions for selecting strongly asymptotic  $\ell_p$  subspaces of a given Banach space were given in [FFKR] and [T].

Our main results are proved in §3 (Theorem 9) and §4 (Theorem 14). We summarize these results in slightly weaker form in

**Theorem 1.** *Let  $1 \leq p \leq \infty$  and let  $X$  be a Banach space with a strongly asymptotic  $\ell_p$  basis  $(e_i)$ :*

- *if  $X$  is minimal and  $1 \leq p < 2$ , then  $X$  is isomorphic to a subspace of  $\ell_p$ ;*
- *if  $X$  is minimal and  $2 \leq p < \infty$ , or if  $X$  is complementably minimal and  $1 \leq p \leq \infty$ , then  $(e_i)$  is equivalent to the unit vector basis of  $\ell_p$  (or  $c_0$  if  $p = \infty$ ).*

For  $1 \leq p < 2$ , we also give examples of strongly asymptotic  $\ell_p$  basic sequences in  $\ell_p$  spanning minimal spaces that are not isomorphic to  $\ell_p$  (Theorem 12).

Recall again that  $T^*$  is a minimal space which is strongly asymptotic  $\ell_\infty$ .

**Question 2.** Does there exist a minimal asymptotic  $\ell_p$  space that does not embed into  $\ell_p$  for  $1 \leq p < \infty$ ?

A broader definition of “asymptotic  $\ell_p$  space” which is independent of a basis is given in [MMT]. The definition is that for some  $C < \infty$  and for all  $n \in \mathbb{N}$ , if  $(e_i)_1^n \in \{X\}_n$ , where  $\{X\}_n$  is the  $n^{\text{th}}$  asymptotic class selected via the filter of finite-codimensional subspaces, then  $(e_i)_1^n$  is  $C$ -equivalent to the unit vector basis of  $\ell_p^n$ . This notion does not seem to lend itself to a

strongly asymptotic version. But these spaces contain asymptotic  $\ell_p$  basic sequences.

We use standard Banach space theory notation and terminology as in [LT1]. All subspaces (of a Banach space  $X$ ) are assumed to be closed and infinite-dimensional unless stated otherwise. Let  $(e_i)$  be a basis for  $X$ . We say that a sequence  $(x_k)$  of nonzero vectors is a *block basis* if there exist integers  $0 = n_0 < n_1 < \dots$  and scalars  $(a_i)$  such that

$$x_k = \sum_{i=n_{k-1}+1}^{n_k} a_i e_i \quad (k \geq 1).$$

The closed linear span of a block basis is called a *block subspace*.

A minimal space is *C-minimal* if every subspace of  $X$  contains a *C-isomorphic copy* of  $X$ , i.e. there exist  $W \subseteq X$  and an isomorphism  $T: X \rightarrow W$  with  $\|T\|\|T^{-1}\| \leq C$ . P. Casazza proved that a minimal space must be *C-minimal* for some  $C < \infty$  [C].

The second author and C. Rosendal [FR2] defined a space  $X$  with a basis  $(e_i)$  to be *block minimal* (resp. *C-block minimal*) if every block subspace of  $X$  has a further block subspace isomorphic (resp. *C-isomorphic*) to  $X$ ; also  $X$  is defined to be *equivalence block minimal* (resp. *C-equivalence block minimal*) if every block basis has a further block basis equivalent (resp. *C-equivalent*) to  $(e_i)$ . They proved that every block minimal (resp. equivalence block minimal) space is *C-block minimal* (resp. *C-equivalence block minimal*) for some  $C < \infty$ ; they deduced that a basis which is asymptotic  $\ell_p$  (in the broader sense associated to the filter of tail subspaces) and equivalence block minimal must be equivalent to the unit vector basis of  $\ell_p$  or  $c_0$ .

We remark that  $T^*$  is minimal but does not contain any block minimal block subspace. Indeed, recall [CS] that any block basis of  $T^*$  is equivalent to a subsequence of the standard basis  $(t_n^*)$  and that the subspaces  $[(t_{n_k}^*)]$  and  $[(t_{m_k}^*)]$  are isomorphic if and only if  $(t_{n_k}^*)$  and  $(t_{m_k}^*)$  are equivalent. Thus, if  $T^*$  had a block minimal block subspace  $[(x_i^*)]$ , then  $[(x_i^*)]$  would be equivalence block minimal and therefore *C-equivalence block minimal* for some  $C < \infty$ . That, combined with  $(x_i^*)$  being an asymptotic  $\ell_\infty$  basis,

would imply that  $(x_i^*)$  is equivalent to the unit vector basis of  $c_0$ , which is a contradiction since  $T^*$  does not contain  $c_0$ .

## 2. ASYMPTOTIC $\ell_p$ SUBSPACES OF $L_p$

Our reference for results concerning  $L_p$  will mostly be the survey article by D. Alspach and the fourth author [AO]. For the definition and properties of the Haar basis we take [LT2, Section 2.c] as our reference.

We note that every strongly asymptotic  $\ell_p$  sequence  $(e_i)$  is unconditional. Indeed, if  $x$  and  $y$  are disjointly supported vectors belonging to  $[(e_i)_{i \geq f(2)}]$ , then

$$\|x \pm y\| \approx (\|x\|^p + \|y\|^p)^{1/p},$$

which implies unconditionality.

Let  $X$  be a Banach space with a basis  $(e_i)$ . A *blocking* of  $(e_i)$  is a finite-dimensional decomposition (FDD) for  $X$  corresponding to a partition  $\{F_n : n \in \mathbb{N}\}$  of  $\mathbb{N}$  into successive finite subsets, i.e.  $X = \sum_{n=1}^{\infty} [e_i : i \in F_n]$ . Let  $T_\omega = \{(n_1, \dots, n_k) \in \mathbb{N}^k : k \in \mathbb{N}\}$  be the countably branching tree ordered by  $(n_1, \dots, n_k) \leq (m_1, \dots, m_l)$  if  $k \leq l$  and  $n_i = m_i$  for  $i \leq k$ . We say that  $T \in T_\omega(X)$  if  $T = \{x_{(n_1, \dots, n_k)} : (n_1, \dots, n_k) \in T_\omega\} \subset S_X$ .  $T$  is a *block tree with respect to the basis  $(e_i)$*  if in addition  $(x_{(n)})_{n \in \mathbb{N}}$  and  $(x_{(n_1, \dots, n_k, n)})_{n \in \mathbb{N}}$  are all block bases of  $(e_i)$  for all  $(n_1, \dots, n_k) \in T_\omega$ .

**Proposition 3.** *Let  $1 < p < \infty$  and let  $X$  be a subspace of  $L_p[0, 1]$  with an asymptotic  $\ell_p$  basis  $(e_i)$ . Then  $X$  embeds into  $\ell_p$ . If  $p \geq 2$  and  $(e_i)$  is strongly asymptotic  $\ell_p$ , then  $(e_i)$  is actually equivalent to the unit vector basis of  $\ell_p$ .*

*Proof.* We may of course assume that  $p \neq 2$ . Then the space  $X$  does not contain a subspace isomorphic to  $\ell_2$ , otherwise some block basis of  $(e_i)$  would be equivalent to the unit vector basis of  $\ell_2$ , contradicting the fact that  $(e_i)$  is asymptotic  $\ell_p$ . If  $p > 2$  this implies by [AO, Theorem 30] that  $X$  embeds into  $\ell_p$  (in fact,  $X$   $1+\varepsilon$ -embeds into  $\ell_p$  for all  $\varepsilon > 0$  [KW]). Assume now that  $1 < p < 2$ . By a theorem of W. B. Johnson [AO, Theorem 30] it suffices to prove that for some  $C < \infty$  every normalized block basis  $(x_i)$  of  $(e_i)$

admits a subsequence  $C$ -equivalent to the unit vector basis of  $\ell_p$ . Passing to a subsequence we may assume that  $(x_i)$  is a perturbation of a block basis of the Haar basis and hence is  $C(p)$ -unconditional for some constant  $C(p)$  depending only on  $p$  [LT2, Theorem 2.c.5]. By [AO, Lemma 28] it suffices to show that there exist  $\delta(K) > 0$ , where  $K$  is the asymptotic  $\ell_p$ -constant of  $(e_i)$ ,  $(y_i) \subseteq (x_i)$  and disjoint measurable sets  $E_i$  in  $[0, 1]$  with  $\|y_i|_{E_i}\| \geq \delta(K)$  for all  $i$ .

By a theorem of Dor [D] there exists  $\delta(K) > 0$  such that if  $(x_i)_{i \in A}$  is  $K$ -equivalent to the unit vector basis of  $\ell_p^{|A|}$  then there exist disjoint measurable sets  $(F_i)_{i \in A}$  in  $[0, 1]$  with  $\|x_i|_{F_i}\| > \delta(K)$  for  $i \in A$ . Since  $(e_i)$  is  $K$ -asymptotic  $\ell_p$  we can thus, for all  $n$  and  $\varepsilon > 0$ , choose  $i > n$  and a measurable set  $F_i$  with  $\lambda(F_i) < \varepsilon$  (where  $\lambda$  denotes Lebesgue measure) and  $\|x_i|_{F_i}\| > \delta(K)$ . We then proceed inductively. Let  $y_1 = x_1$  and  $F_1^1 = [0, 1]$ . Assume that  $(y_1, \dots, y_n)$  have been chosen along with disjoint sets  $F_1^n, \dots, F_n^n$  so that  $\|y_i|_{F_i^n}\| > \delta(K)$ . Applying the above for a suitable  $\varepsilon$ , using the uniform integrability of  $(|y_i|^p)_{i=1}^n$ , we can choose  $y_{n+1}$  and  $F_{n+1}^{n+1}$  so that  $\|y_{n+1}|_{F_{n+1}^{n+1}}\| > \delta(K)$ ,  $(y_i)_{i=1}^{n+1}$  is a subsequence of  $(x_i)$ , and  $\|y_i|_{F_i^{n+1}}\| > \delta(K)$  for  $i \leq n$ , where  $F_i^{n+1} = F_i^n \setminus F_{n+1}^{n+1}$ . The claim now follows by setting  $E_i = \bigcap_{n \geq i} F_i^n$ .

Finally, assume that  $p > 2$  and that  $(e_i)$  is strongly asymptotic  $\ell_p$  with constant  $C < \infty$  and an associated function  $f$ . As we noted,  $(e_i)$  is unconditional. By [DS] (see also [JMST] or [LT2, Corollary 2.e.19] for a more general lattice version) a normalized unconditional sequence  $(x_i)$  in  $L_p$  is either equivalent to the unit vector basis of  $\ell_p$ , or it has, for all  $n \in \mathbb{N}$ ,  $n$  disjoint normalized blocks which form a sequence 2-equivalent to the unit vector basis of  $l_2^n$ .

Let  $k \in \mathbb{N}$  be such that the unit vector bases of  $l_2^k$  and  $l_p^k$  are not  $2C$ -equivalent. Since  $(e_i)$  is strongly asymptotic  $\ell_p$ , there cannot exist a normalized length  $k$  sequence of disjoint blocks on  $(e_i)_{i \geq f(k)}$  which is 2-equivalent to the basis of  $l_2^k$ . So  $(e_i)_{i \geq f(k)}$ , and therefore  $(e_i)$ , is equivalent to the unit vector basis of  $\ell_p$ .  $\square$

Note that it remains open whether we may remove “strongly” from the last part of Proposition 3.

**Question 4.** Does there exist an unconditional asymptotic  $\ell_p$  basic sequence in  $L_p$ ,  $p > 2$ , which is not equivalent to the unit vector basis of  $\ell_p$ ?

**Proposition 5.** *Let  $X$  be a subspace of  $L_1[0, 1]$ . If  $X$  has an unconditional asymptotic  $\ell_1$  basis then  $X$  embeds into  $\ell_1$ .*

*Proof.* Let  $(e_i)$  be an unconditional asymptotic  $\ell_1$  basis for  $X$ . Then  $(e_i)$  is necessarily boundedly complete, so by [AO, Proposition 31] it suffices to prove that for some  $C < \infty$  if  $T \in T_\omega(X)$  is a block tree with respect to  $(e_i)$  then some branch of  $T$  is  $C$ -equivalent to the unit vector basis of  $\ell_1$ . The proof of this is much like the previous proof in the case  $1 < p < 2$ . We need only show that there exists  $\delta = \delta(K) > 0$  so that for all such  $T$  we can find a branch  $(y_i)$  and disjoint measurable sets  $(E_i)$  with  $\|y_i|_{E_i}\| > \delta$  for all  $i$ . Again we use Dor’s theorem. If  $(y_i)_1^n$  have been chosen as an initial segment in  $T$  along with disjoint sets  $(F_i)_{i=1}^n$  satisfying  $\|y_i|_{F_i}\| > \delta$  for  $i \leq n$  then we can select  $y_{n+1}$  from among the successors of  $y_n$  and  $F_{n+1}^{n+1}$  with  $\|y_{n+1}|_{F_{n+1}^{n+1}}\| > \delta$  and  $\lambda(F_{n+1}^{n+1})$  as small as we please. In particular we may insure that  $\|y_i|_{F_i^n \setminus F_{n+1}^{n+1}}\| > \delta$  for  $i \leq n$ . Then as earlier we let  $E_i = \cap_{n \geq i} F_i^n$ .  $\square$

**Remarks 6.** (1) We do not know if Proposition 5 holds if we merely assume that  $(e_i)$  is asymptotic  $\ell_1$ . Then  $(e_i)$  would still be boundedly complete but we need unconditionality to conclude that the branch  $(y_i)$  with  $\|y_i|_{E_i}\| > \delta$  is  $C(K)$ -equivalent to the unit vector basis of  $\ell_1$  using [AO, Lemma 28].

However, we can drop “unconditionality” in any of the following three cases:

- (a)  $(e_i)$  is a block basis of  $(h_i)$  or more generally for all  $n \in \mathbb{N}$

$$\lim_{m \rightarrow \infty} \sup \{ \|P_n x\| : x \in B_{[e_i]_{i=m}^\infty} \} = 0,$$

where  $P_n$  is the basis projection of  $L_1[0, 1]$  onto the span of the first  $n$  Haar functions;

- (b) for some  $K < 2$  every normalized block basis of  $(e_i)$  admits a subsequence  $K$ -equivalent to the unit vector basis of  $\ell_1$ .

(c)  $(e_i)$  is  $K$ -asymptotic  $\ell_1$  with  $K < 2$ .

Indeed, the branch  $(y_i)$  obtained in the proof can be chosen so that

$$\left\| \sum_i a_i y_i \right\| \geq \left\| \sum_i a_i y_{i|_{\cup_j E_j}} \right\| \gtrsim \sum_i \left\| a_i y_{i|_{E_i}} + \sum_{j>i} a_j y_{j|_{E_i}} \right\|.$$

Now under (a) we can also choose  $(y_i)$  so that, for all  $i$ ,  $(y_{j|_{E_i}})_{j \geq i}$  is 2-basic and so

$$\left\| a_i y_{i|_{E_i}} + \sum_{j>i} a_j y_{j|_{E_i}} \right\| \geq \frac{1}{2} |a_i| \|y_{i|_{E_i}}\| \geq \frac{1}{2} |a_i| \delta.$$

Next we shall show that case (b) yields that we can choose the branch  $(y_i)$  and sets  $(E_i)$  so that  $\|y_{i|_{E_i}}\| > \delta > 1/2$  if  $1/2 < \delta < 1/K$ . Since

$$\begin{aligned} \sum_i \left\| a_i y_{i|_{E_i}} + \sum_{j>i} a_j y_{j|_{E_i}} \right\| &\geq \sum_i \left( |a_i| \delta - \sum_{j>i} |a_j| \|y_{j|_{E_i}}\| \right) \\ &\geq \delta \sum_i |a_i| - \sum_i |a_i| \|y_{i|[0,1] \setminus E_i}\| \\ &\geq (\delta - (1 - \delta)) \sum_i |a_i| = (2\delta - 1) \sum_i |a_i|, \end{aligned}$$

we obtain that  $(y_i)$  is equivalent to the unit vector basis of  $\ell_1$ .

Suppose that  $(x_i)$  is a normalized block basis of  $(e_i)$  and (b) holds. Passing to a subsequence we may assume that  $(x_i)$  is  $K$ -equivalent to the unit vector basis of  $\ell_1$ . Moreover by Rosenthal's "subsequence splitting lemma" (see e.g. [GD] or [JMST, p. 135] for a general lattice version) we may assume that  $(x_i) = (u_i + z_i)$  where  $(z_i) = (x_{i|_{E_i}})$  is disjointly supported,  $\lim_i \|z_i\| = b$ , and  $(|u_i|)$  is uniformly integrable. It suffices to prove that  $b \geq 1/K$ . For then the proof of Proposition 5 will yield the  $\|y_{i|_{E_i}}\| > \delta$  claim. Given  $\varepsilon > 0$  we can form an absolute convex combination  $\sum_i a_i x_i$  with  $\|\sum_i a_i u_i\| < \varepsilon$ , and thus  $\|\sum_i a_i z_i\| = \sum_i |a_i| \|z_i\| \geq 1/K - \varepsilon$ . Since these can be found arbitrarily far out we deduce that  $b \geq 1/K$ .

Finally assume (c) holds. Let  $(x_i)$  again be a normalized block basis of  $(e_i)$ . We may assume, passing to a subsequence, as in case (b) that  $(x_i) = (u_i + z_i)$ ,  $(z_i) = (x_{i|_{E_i}})$  is disjointly supported,  $\lim_i \|z_i\| = b$ ,  $(|u_i|)$  is uniformly integrable, and moreover  $u_i = u + v_i$  where  $(v_i)$  is a weakly null perturbation of a block basis of the Haar basis for  $L_1[0, 1]$ . We assume  $\|v_i\| \rightarrow a \in [0, 2]$  and first assume  $a \neq 0$ . Passing again to a subsequence we

may assume  $(v_i)$  has a spreading model  $(\tilde{v}_i)$  which is unconditional, subsymmetric, and (e.g. by Dor's theorem [D]) is not equivalent to the unit vector basis of  $\ell_1$ . Thus  $\lim_n (1/n) \|\sum_{i=1}^n \tilde{v}_i\| = 0$  (see e.g. [BL]). In particular, we can thus find (even if  $a = 0$ ) for all  $\varepsilon > 0$  an absolute convex combination  $\sum_{i \in E_\varepsilon} a_i^\varepsilon x_i$  with  $\|\sum_{i \in E_\varepsilon} a_i^\varepsilon u_i\| < \varepsilon$  and  $f(|E_\varepsilon|) \leq \min E_\varepsilon$ , where  $f$  is the function associated to the asymptotic  $\ell_1$  sequence  $(e_i)$ . As in case (b) we deduce that  $b \geq 1/K$  and finish the proof as in case (b).

(2) Note that if  $(e_i)$  is a  $K$ -asymptotic  $\ell_1$  basis for  $X \subset L_1[0, 1]$  then  $X$  has the strong Schur property. To see this, let  $\delta > 0$  and let  $(x_i)$  be a sequence in  $B_X$  satisfying  $\|x_i - x_j\| \geq \delta$ . Since  $(e_i)$  is boundedly complete, we may assume (by passing to a subsequence) that  $x_i = y + z_i$ , where  $y, z_i \in X$  and  $(z_i)$  is a perturbation of a block basis of  $(e_i)$ . We may assume that  $\|z_i\| \rightarrow a \geq \delta/2$  and that  $(z_i/\|z_i\|)$  is  $K'$ -asymptotic  $\ell_1$  where  $K' > K$  is arbitrary. As in (1) we may also assume, by passing to a subsequence, that there exist disjoint sets  $(E_i)$  with  $\lim_i \|(z_i/\|z_i\|)|_{E_i}\| = b \geq 1/K'$  and so  $\lim_i \|z_i|_{E_i}\| = ab \geq \delta/(2K')$ . By Rosenthal's lemma [R] some subsequence of  $(z_i)$  is  $(4K'/\delta)$ -equivalent to the unit vector basis of  $\ell_1$ . Hence a subsequence of  $(x_i)$  is  $(4K'(B+2)/\delta)$ -equivalent to the unit vector basis of  $\ell_1$ , where  $B$  is the basis constant of  $(e_i)$ .

Bourgain and Rosenthal constructed a strong Schur subspace of  $L_1[0, 1]$  with a basis which does not embed into  $\ell_1$  [BR]. That space fails the Radon-Nikodým property and hence does not have a boundedly complete basis. In particular, it does not have an asymptotic  $\ell_1$  basis.

(3) Let  $(E_i)$  be an FDD for a Banach space  $X$ . We say that  $(E_i)$  is an asymptotic  $\ell_p$  (resp. strongly asymptotic  $\ell_p$ ) FDD if there exist  $C < \infty$  and an increasing function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that, for all  $n \in \mathbb{N}$ , every normalized block basis (resp. sequence of disjointly supported vectors)  $(x_i)_{i=1}^n$  of  $[(E_i)_{i=f(n)}^\infty]$  is  $C$ -equivalent to the unit vector basis of  $\ell_p^n$ . The proof of Proposition 3 carries over to show that if  $X$  is a subspace of  $L_p$  ( $1 < p < \infty$ ) with an asymptotic  $\ell_p$  FDD then  $X$  embeds into  $\ell_p$ . Similarly, Proposition 5 remains valid if "basis" is replaced by "FDD".

3. STRONGLY ASYMPTOTIC  $\ell_p$  SPACES AND PRIMARY MINIMALITY

A key ingredient of the proof of Theorem 9 is a uniformity lemma about embedding into subspaces generated by tail subsequences of a basis  $(e_i)_{i \in \mathbb{N}}$  (i.e. subsequences of the form  $(e_i)_{i \geq k}$ , for  $k \in \mathbb{N}$ ).

We recall that a Banach space  $X$  is said to be *primary* if whenever  $X = Y \oplus Z$ , then either  $X$  is isomorphic to  $Y$  or  $X$  is isomorphic to  $Z$ .

**Definition 7.** A Schauder basis  $(e_n)_{n \in \mathbb{N}}$  is said to be *primarily minimal* if whenever  $(I, J)$  is a partition of  $\mathbb{N}$ , then either  $[e_n]_{n \in \mathbb{N}}$  embeds into  $[e_n]_{n \in I}$  or  $[e_n]_{n \in \mathbb{N}}$  embeds into  $[e_n]_{n \in J}$ . A Banach space  $X$  is *primarily minimal* if whenever  $X = Y \oplus Z$ , then either  $X$  embeds into  $Y$  or  $X$  embeds into  $Z$ .

Obviously any unconditional basis of a primarily minimal space must be primarily minimal. Also the class of primarily minimal spaces contains the class of primary spaces as well as the class of minimal spaces.

**Proposition 8.** *Let  $X$  be a Banach space with a primarily minimal basis  $(e_n)_{n \in \mathbb{N}}$ . Then there exists  $C < \infty$  such that  $X$   $C$ -embeds into all of its tail subspaces.*

*Proof.* Let  $2^\omega$  be equipped with its usual topology, i.e. basic open sets are of the form  $N(u) = \{\alpha = (\alpha_k)_{k \in \mathbb{N}} \in 2^\omega : \forall k \leq m, \alpha_k = u_k\}$ , where  $u = (u_k)_{k \leq m}$  is a sequence of 0's and 1's of length  $m$ .

Let  $A$  be the set of  $\alpha$ 's in  $2^\omega$  such that  $X$  embeds into  $[e_i]_{\alpha_i=1}$ , and let  $A'$  be the set of  $\alpha$ 's in  $2^\omega$  such that  $X$  embeds into  $[e_i]_{\alpha_i=0}$ . We also let, for  $n \in \mathbb{N}$ ,  $A_n$  (resp.  $A'_n$ ) be the set of  $\alpha$ 's such that  $X$   $n$ -embeds into  $[e_i]_{\alpha_i=1}$  (resp.  $[e_i]_{\alpha_i=0}$ ).

Now  $(e_n)$  being primarily minimal means that  $2^\omega = A \cup A' = (\cup_{n \in \mathbb{N}} A_n) \cup (\cup_{n \in \mathbb{N}} A'_n)$ . By the Baire Category Theorem we deduce that the closure of  $A_N$  or  $A'_N$  has nonempty interior for some  $N \in \mathbb{N}$ , and hence  $A_N$  or  $A'_N$  is dense in some basic neighborhood. We note that the map  $f : 2^\omega \rightarrow 2^\omega$  defined by  $f((\alpha_i)_{i \in \mathbb{N}}) = (1 - \alpha_i)_{i \in \mathbb{N}}$  is a homeomorphism such that  $f(A_N) =$

$A'_N$ . It follows that  $A_N$  is dense in some basic neighborhood. Let  $u = (u_i)_{i \leq m}$  be such that  $A_N$  is dense in  $N(u)$ .

Let  $E = \{i : u_i = 1, i \leq m\}$ . It follows that  $X$   $N$ -embeds into  $[(e_i)_{i \in E} \cup (e_i)_{i > k}]$  for all  $k \in \mathbb{N}$ . Using the uniform equivalence between  $(e_i)_{i \in E}$  and  $(e_i)_{i \in F}$  for all  $|F| = |E|$ , we see that there exists  $C$  such that  $X$   $C$ -embeds into  $[(e_i)_{i > k}]$  for all  $k$ .  $\square$

**Theorem 9.** *Let  $X$  have a primarily minimal, strongly asymptotic  $\ell_p$  basis  $(e_i)$  for some  $1 \leq p < \infty$ . If  $1 \leq p < 2$  then  $X$  embeds into  $\ell_p$ , and if  $p \geq 2$  then  $(e_i)$  is equivalent to the unit basis of  $\ell_p$ .*

*Proof.* We may assume that  $X$  has a 1-unconditional basis  $(e_i)$  with a strongly asymptotic  $\ell_p$  constant  $C$ .

By Proposition 8 there exists  $K < \infty$  such that  $X$   $K$ -embeds into each of its tail subspaces associated to  $(e_n)_{n \in \mathbb{N}}$ . By a result of Johnson [J], for each  $n$  there exists  $N(n)$  such that for any  $n$ -dimensional subspace  $F$  of any Banach space  $E$  with a 1-unconditional basis there exist  $N(n)$  normalized disjointly supported vectors  $(w_i)_{i=1}^{N(n)}$  (with respect to the basis) such that  $F$  is 2-isomorphic to a subspace of  $[(w_i)_{i=1}^{N(n)}]$ . Let  $F$  be an arbitrary finite-dimensional subspace of  $X$  and let  $n$  be its dimension. Consider  $Y = [(e_i)_{i \geq f(N(n))}]$ . Then  $Y$  contains a  $K$ -isomorphic copy of  $X$ , and thus also of  $F$ . By the above result, there exist  $N = N(n)$  normalized disjointly supported vectors  $(w_i)_{i=1}^N$  of  $[(e_i)_{i \geq f(N)}]$  such that  $F$  is  $2K$ -isomorphic to a subspace of  $W := [(w_i)_{i=1}^N]$ . Therefore,  $W$  is  $C$ -isomorphic to  $\ell_p^N$ . Thus,  $F$  is  $2CK$ -isomorphic to a subspace of  $\ell_p^N$ . It follows that  $X$  is crudely finitely representable in  $\ell_p$  and hence embeds isomorphically into  $L_p[0, 1]$  [LP]. The result now follows from Proposition 3 and Proposition 5.  $\square$

**Remark 10.** For  $1 \leq p < 2$ , we actually obtain that  $(e_i)$  may be blocked into a decomposition  $\sum_{i=1}^{\infty} \oplus F_i \cong (\sum_{i=1}^{\infty} \oplus F_i)_p$ , where the  $F_i$ 's embed uniformly into  $\ell_p$ . Indeed, any unconditional basis for a subspace  $X$  of  $\ell_p$ ,  $1 \leq p < \infty$ , may be blocked into such an FDD. For  $p > 1$  this is due to W. B. Johnson and M. Zippin [JZ], and unconditionality is not required. For  $p = 1$ , this follows for example from [AO, Proposition 31].

As we mentioned in the Introduction, we do not know if a positive result similar to Theorem 1 is true for the more general class of asymptotic  $\ell_p$  spaces,  $1 \leq p < \infty$ . For example, we do not know what the case is with the Argyros-Deliyanni mixed Tsirelson space  $X_u$  [AD]. Two main ingredients were used in [CJT] to prove the minimality of  $T^*$ , namely the universality of  $\ell_\infty^n$ 's for all finite dimensional spaces and an appropriate blocking principle. For  $X_u$  there is no corresponding blocking principle. On the other hand, it was proved in [ADKM] that all of its subspaces contain uniformly  $\ell_\infty^n$ 's, which makes it impossible to use a local argument as in the case of strongly asymptotic  $\ell_p$  spaces.

Note that there is no version of Theorem 9 for spaces with a primarily minimal FDD (which is defined by replacing the basis  $(e_i)$  by an FDD  $(E_i)$  throughout Definition 7). Indeed, let  $X = (\sum_{n=1}^\infty \oplus \ell_\infty^n)_p$ , where  $1 \leq p < \infty$ . Then the natural FDD for  $X$  is easily seen to be a primarily minimal strongly asymptotic  $\ell_p$  FDD, but obviously  $X$  does not embed into  $\ell_p$ . However, we have the following result for minimal spaces.

**Proposition 11.** *Suppose that  $X$  is minimal and that  $(E_i)$  is a strongly asymptotic  $\ell_p$  FDD for  $X$ , where  $1 \leq p < \infty$ . Then  $X$  embeds into  $\ell_p$ .*

*Proof.* It suffices to show that  $\ell_p$  embeds into  $X$ . Choose  $e_i \in E_i$  for  $i \geq 1$  with  $\|e_i\| = 1$ . Then  $(e_i)$  is a strongly asymptotic  $\ell_p$  sequence spanning a minimal subspace  $Y$  of  $X$ , so  $Y$  embeds into  $\ell_p$  by Theorem 9, and hence  $\ell_p$  embeds into  $X$ .  $\square$

We now present examples of minimal strongly asymptotic  $\ell_p$  spaces that are not isomorphic to  $\ell_p$ , for  $1 \leq p < 2$ . Note that if  $(e_i)$  is a strongly asymptotic  $\ell_p$  basis ( $1 \leq p \leq \infty$ ), then  $(e_i^*)$  is a strongly asymptotic  $\ell_q$  ( $1/p + 1/q = 1$ ) basic sequence with the same  $f : \mathbb{N} \rightarrow \mathbb{N}$ . This follows easily from the unconditionality of  $(e_i)$  and a standard Hölder's inequality calculation.

**Theorem 12.** *Let  $1 \leq p < 2$ . There exists a minimal Banach space  $X$  with a strongly asymptotic  $\ell_p$  basis  $(e_i)$  satisfying the following:*

- (i)  $X$  is not isomorphic to  $\ell_p$ ;
- (ii) If  $p > 1$  then  $X^*$  does not embed into  $L_q$  ( $1/p + 1/q = 1$ ).

*Proof.* We shall use the following two facts (the first follows from [KP], and the second follows from the existence, due to Paul Lévy, of  $s$ -stable random variables for  $1 < s < 2$ ): firstly, if  $s \notin \{2, p\}$  and  $M > 0$  then there exists  $N$  such that  $\ell_p$  does not contain an  $M$ -complemented  $M$ -isomorphic copy of  $\ell_s^N$ ; secondly, if  $p < s < 2$  then  $\ell_p$  contains almost isometric copies (i.e.  $(1 + \varepsilon)$ -isomorphic copies for all  $\varepsilon > 0$ ) of  $\ell_s^n$  for all  $n$ . For each  $n$ , let  $s_n$  be defined by the equation  $1/s_n := 1/p - (1/p - 1/2)/(2n)$ . Note that  $p < s_n < 2$  and that  $s_n \rightarrow p$  rapidly enough to ensure that the standard bases of  $\ell_{s_n}^n$  and  $\ell_p^n$  are 2-equivalent (as is easily checked). By the first fact applied to  $M = n$  and  $s = s_n$ , there exists  $N(n)$  such that  $X_n := \ell_{s_n}^{N(n)}$  is not  $n$ -isomorphic to an  $n$ -complemented subspace of  $\ell_p$ . Let  $X := (\sum_{n=1}^{\infty} \oplus X_n)_p$  and let  $(e_i)$  be the basis of  $X$  obtained by concatenation of the standard bases of  $X_1, X_2, \dots$  in that order.

By the second fact each  $X_n$  is almost isometric to a subspace of  $\ell_p$ , so  $X$  is also almost isometric to a subspace of  $\ell_p$ . Since  $X_n$  is 1-complemented in  $X$  but is not  $n$ -isomorphic to an  $n$ -complemented subspace of  $\ell_p$ , it follows that  $X$  is not isomorphic to  $\ell_p$ , which proves (i). Property (ii) will follow from (i) and the fact that  $(e_i)$  is a strongly asymptotic  $\ell_p$  basis: then  $X^*$  is a strongly asymptotic  $\ell_q$  space which is not isomorphic to  $\ell_q$ , and therefore does not embed into  $L_q$  by Proposition 3.

Finally, we verify that  $(e_i)$  is a strongly asymptotic  $\ell_p$  basis. Suppose that  $n \geq 1$  and that the vectors  $x_1, \dots, x_n$  are disjointly supported vectors (with respect to  $(e_i)$ ) which belong to the tail space  $(\sum_{j=n}^{\infty} \oplus X_j)_p$ . Write

$x_i = \sum_{j=n}^{\infty} x_i^j$ , where  $x_i^j \in X_j$ . Then

$$\begin{aligned} \left\| \sum_{i=1}^n x_i \right\|^p &= \sum_{j=n}^{\infty} \left\| \sum_{i=1}^n x_i^j \right\|^p \\ &= \sum_{j=n}^{\infty} \left( \sum_{i=1}^n \|x_i^j\|^{s_j} \right)^{p/s_j} \end{aligned}$$

(by the disjointness of  $x_1^j, \dots, x_n^j$ )

$$\approx \sum_{j=n}^{\infty} \sum_{i=1}^n \|x_i^j\|^p$$

(by the 2-equivalence of the standard bases of  $\ell_p^n$  and  $\ell_{s_j}^n$  for all  $j \geq n$ )

$$\begin{aligned} &= \sum_{i=1}^n \sum_{j=n}^{\infty} \|x_i^j\|^p \\ &= \sum_{i=1}^n \|x_i\|^p. \end{aligned}$$

□

#### 4. STRONGLY ASYMPTOTIC $\ell_p$ -SPACES AND COMPLEMENTATION

Let us define a basis  $(e_n)_{n \in \mathbb{N}}$  to be *primarily complementably minimal* if for any partition  $(I, J)$  of  $\mathbb{N}$ ,  $[e_n]_{n \in \mathbb{N}}$  is isomorphic to a complemented subspace of  $[e_n]_{n \in I}$  or of  $[e_n]_{n \in J}$ . A space  $X$  is *primarily complementably minimal* if whenever  $X = Y \oplus Z$ , then either  $X$  embeds complementably into  $Y$  or  $X$  embeds complementably into  $Z$ .

Every Banach space which is primary, or which is complementably minimal, is primarily complementably minimal. We now prove a theorem which yields the complementably minimal part of Theorem 1.

Recall that an unconditional basis  $(x_n)_{n \in \mathbb{N}}$  of a Banach space is *sufficiently lattice-euclidean* if it has, for some  $C \geq 1$  and every  $n \in \mathbb{N}$ , a  $C$ -complemented,  $C$ -isomorphic copy of  $\ell_2^n$  whose basis is disjointly supported on  $(x_n)_{n \in \mathbb{N}}$ . See [CK] for a general definition in the lattice setting. Note that strongly asymptotic  $\ell_p$  bases for  $p \neq 2$  are not sufficiently lattice-euclidean.

**Proposition 13.** *Let  $X$  be a Banach space with an unconditional, primarily complementably minimal basis  $(e_n)_{n \in \mathbb{N}}$ . Then there exists  $K < \infty$  such that  $X$   $K$ -embeds as a  $K$ -complemented subspace of its tail subspaces.*

We skip the proof since it is exactly the same as the proof of Proposition 8, *mutatis mutandis*. Note that unconditionality is required to preserve complemented embeddings with uniform constants in the end of the argument.

**Theorem 14.** *Let  $X$  be a primarily complementably minimal Banach space with a strongly asymptotic  $\ell_p$  basis  $(e_n)$ ,  $1 \leq p \leq \infty$ . Then  $(e_n)$  is equivalent to the unit vector basis of  $\ell_p$  (or  $c_0$  if  $p = \infty$ ).*

*Proof.* We may assume that  $p \neq 2$  and we may also assume that  $(e_n)$  is 1-unconditional. By Proposition 13, there exists  $K < \infty$  such that  $X$  is  $K$ -isomorphic to some  $K$ -complemented subspace of any tail subspace  $Y$  of  $X$ . Since  $p \neq 2$ ,  $X$  is not sufficiently lattice euclidean; note also that the canonical basis of  $Y$  is 1-unconditional. Therefore we deduce from [CK, Theorem 3.6] that  $(e_n)$  is  $c$ -equivalent to a sequence of disjointly supported vectors in  $Y^N$ , for some  $c$  and  $N$  depending only on  $X$  and  $K$ . Here  $Y^N$  is equipped with the norm  $\|(y_i)_{i=1}^N\| = \max_{1 \leq i \leq N} \|y_i\|$  and with the canonical basis obtained from the basis  $(b_i)$  of  $Y$  with the ordering  $(b_1, 0, \dots, 0), (0, b_1, 0, \dots, 0), (0, \dots, 0, b_1), (b_2, 0, \dots, 0)$ , etc. It is easy to check that  $Y^N$  is also strongly asymptotic  $\ell_p$  with constant  $C$  and function  $f$  depending only on  $N$  and on the strongly asymptotic  $\ell_p$  constant and function of  $(e_n)$ .

Let  $k \in \mathbb{N}$  be arbitrary and let  $Y$  be a tail subspace of  $X$  such that  $Y^N$  is supported after the  $f(k)$ -th vector of the basis of  $X^N$ . The sequence  $(e_i)_{i \leq k}$  is therefore  $c$ -equivalent to a sequence which is disjointly supported after  $f(k)$ , so is  $cC$ -equivalent to the unit vector basis of  $\ell_p^k$ .  $\square$

**Remark 15.** We applied [CK, Theorem 3.6] with  $E_n = \mathbb{N}$  for all  $n$ . There is an inaccuracy in the statement of the theorem: the sets  $E_n$  do not need to be assumed disjoint. Theorem 3.6 is based on [CK, Lemma 3.3], and we want to point out an imprecision in the proof of Lemma 3.3. The inequality on p.150,

line 11, goes the opposite way. In the special case in which we apply Theorem 3.6, however, we have  $h_n = |Se_n||T^*e_n|$  and the proof is clear. Actually, as Nigel Kalton has informed us, Lemma 3.3 is true as stated, but one needs a small adjustment in the general case. More precisely, one should define the functions  $f_n$  and  $g_n$  in a slightly different way. If  $0 \leq h_n \leq |Se_n||T^*e_n|$ , we have that  $h_n$  can be expressed in the form  $h_n = f_n g_n$ , where  $|f_n| \leq |Se_n|$  and  $|g_n| \leq |T^*e_n|$ . Then the proof is correct as it stands.

## 5. SOME CONSEQUENCES ABOUT THE NUMBER OF NONISOMORPHIC SUBSPACES OF A BANACH SPACE

The question of the number of mutually nonisomorphic subspaces of a Banach space which is not isomorphic to  $\ell_2$  was first investigated by the second author and Rosendal [FR1], [FR2], [R2] (but some ideas originated from an earlier paper of N. J. Kalton [K2]). A separable Banach space  $X$  is said to be *ergodic* if the relation  $E_0$  of eventual agreement between sequences of 0's and 1's is Borel reducible to isomorphism between subspaces of  $X$ ; this means that there exists a Borel map  $f$  mapping elements of  $2^\omega$  to subspaces of  $X$  such that  $\alpha E_0 \beta$  if and only if  $f(\alpha) \simeq f(\beta)$ . We refer to [FR2] or [R2] for detailed definitions. We just note here that an ergodic Banach space  $X$  must contain  $2^\omega$  mutually nonisomorphic subspaces, and furthermore, that it admits no Borel classification of isomorphism classes by real numbers, i.e. no Borel map  $f$  mapping subspaces of  $X$  to reals, with  $Y \simeq Z$  if and only if  $f(Y) = f(Z)$ . A natural conjecture is to ask if any Banach space nonisomorphic to  $\ell_2$  must be ergodic.

**Theorem 16.** *Let  $1 \leq p \leq \infty$  and let  $X$  be a Banach space with a strongly asymptotic  $\ell_p$  basis  $(e_i)$ . Then  $(e_i)$  is equivalent to the unit vector basis of  $\ell_p$  (or  $c_0$  if  $p = \infty$ ), or  $E_0$  is Borel reducible to isomorphism between subspaces of  $X$  spanned by subsequences of the basis (and in particular there are continuum many mutually nonisomorphic complemented subspaces of  $X$ ).*

*Proof.* The basis  $(e_n)$  is unconditional. If  $E_0$  is not Borel reducible to isomorphism between subspaces of  $X$  spanned by subsequences of the basis, then

by [FR1], [R2], there exists  $K < \infty$  such that the set  $\{\alpha : [e_n]_{\alpha_n=1} \simeq^K X\}$  is comeager in  $2^\omega$ . In particular the set  $A_K$  of  $\alpha$ 's such that  $X$   $K$ -embeds  $K$ -complementably in  $[e_n]_{\alpha_n=1}$  is comeager, thus dense, and the proof of Proposition 13 applies to deduce that for some  $K' < \infty$ ,  $X$  is  $K'$ -isomorphic to some  $K'$ -complemented subspace of any tail subspace of  $X$ . Then the proof of Theorem 14, if  $p \neq 2$ , or Theorem 9, if  $2 \leq p < \infty$ , applies.  $\square$

A consequence of this result is that the versions  $T_p$  of Tsirelson's space are ergodic for  $1 < p < \infty$ . For  $T$ , a stronger result was already proved by Rosendal [R1]. Another consequence is that the mixed Tsirelson spaces and their  $p$ -convexifications for  $1 < p < \infty$  [ADKM] are also ergodic. For a space  $X$  with a strongly asymptotic  $\ell_p$  FDD  $(F_i)$ , we deduce from Theorem 16 that  $X$  is ergodic, or that  $X = \sum \oplus F_i \simeq (\sum \oplus F_i)_p$  in which case  $X \simeq \ell_p(X)$  by [FG2] Corollary 2.12 (with the usual modifications when  $p = \infty$ ).

**Corollary 17.** *Let  $1 \leq p \leq \infty$  and let  $X$  be a Banach space with a strongly asymptotic  $\ell_p$  basis. Then  $X$  is isomorphic to  $\ell_2$  or  $X$  contains  $\omega_1$  nonisomorphic subspaces.*

*Proof.* Assume  $X$  contains no more than countably many mutually nonisomorphic subspaces. By the above,  $X$  is isomorphic to  $\ell_p$  (or  $c_0$  if  $p = \infty$ ). It is known that for  $p \neq 2$ ,  $\ell_p$  contains at least  $\omega_1$  nonisomorphic subspaces [LT1, FG1] (in fact,  $c_0$  and  $\ell_p$ ,  $1 \leq p < 2$ , are ergodic). So  $X$  is isomorphic to  $\ell_2$ .  $\square$

**Corollary 18.** *Let  $1 \leq p \leq \infty$ . Let  $X$  be a Banach space with a strongly asymptotic  $\ell_p$  FDD. Then  $X$  is isomorphic to  $\ell_2$  or  $X$  contains  $\omega$  nonisomorphic subspaces.*

*Proof.* By the above we may assume that  $X \simeq \ell_2(X)$ . If  $X$  has finite cotype and  $X$  is not isomorphic to  $\ell_2$ , then by [A],  $\ell_2(X)$  and therefore  $X$  contains at least  $\omega$  nonisomorphic subspaces. If  $X$  does not have finite cotype, it contains  $\ell_\infty^n$ 's uniformly and therefore  $X$  contains copies of the space  $Y_p = (\sum_{n \in \mathbb{N}} \ell_p^n)_2$  for any  $p \in [1, \infty]$ . For  $p > 2$  it is easy to check that  $\sum_{i=1}^k \|y_i\| \leq k^{1/p'} \|\sum_{i=1}^k y_i\|$  for any disjointly supported  $y_1, \dots, y_k$  on

the canonical basis of  $Y_p$  (see e.g. [FG1, Lemma 2.4]). Therefore by [K1, Lemma 9.3]  $Y_p$  satisfies a lower  $r$ -estimate for any  $r > p$  and therefore has cotype  $r$  [LT2] p.88. On the other hand  $Y_p$  contains  $\ell_p^n$ 's uniformly and therefore does not have cotype  $q$  for  $q < p$ . Therefore the spaces  $Y_p$ ,  $p > 2$ , are mutually nonisomorphic.  $\square$

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