THE WEAK CHEBYSHEV X-GREEDY ALGORITHM IN THE UNWEIGHTED BERGMAN SPACE

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ABSTRACT. We study a greedy algorithm called the Weak Chebyshev X-Greedy Algorithm (WCXGA) and investigate its application to unweighted Bergman spaces. We first show that the WCXGA converges for a wide class of real and complex Banach spaces and dictionaries. We then prove that certain Bergman spaces and their holomorphic monomial dictionaries belong to the class of Banach spaces for which the WCXGA converges.

1. INTRODUCTION

The study of greedy algorithms in Banach spaces—algorithms which allow nonlinear m-term approximation from sets of functions called dictionaries—has become widespread in recent years. Using a dictionary to approximate functions is different from using a basis since a dictionary often contains redundancy. However, approximation with arbitrary dictionaries is currently not as well understood as approximation with bases.

In this paper, we study a specific greedy algorithm called the Weak Chebyshev X-Greedy Algorithm (WCXGA) and investigate its application to unweighted Bergman spaces. We first show that the WCXGA converges for a wide class of real and complex Banach spaces and dictionaries. We then show that certain Bergman spaces and their holomorphic monomial dictionaries belong to the class of Banach spaces for which the WCXGA converges. A particular result is the following: if f is an integrable, holomorphic function on the unit disk in $\mathbb{C}$, then the WCXGA approximation of f using the dictionary $\{z^n\}$ converges to f in norm. On the other hand, it is well-known that Taylor series do not necessarily converge in norm in the simplest Bergman space on the open unit disk in $\mathbb{C}$ [22].

Key words and phrases. greedy algorithm, Bergman space.

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Bergman spaces. We define the Bergman spaces as follows. Let U be a bounded domain in $\mathbb{C}^n$, and let λ be normalized Lebesgue measure on U, so that $\lambda(B) = 1$, where B is the Euclidean ball in $\mathbb{C}^n$. Let $\mathcal{H}(U)$ denote the set of holomorphic functions on U, and let $L^1(U)$ denote the set of integrable functions on U with respect to λ. The unweighted Bergman space is $\mathcal{H}(U) \cap L^1(U)$—the set of holomorphic, integrable functions on U. Throughout, we will let X denote $\mathcal{H}(U) \cap L^1(U)$. The space X is a complex Banach space with norm $\|f\| = \int_U |f| d\lambda$.

The Weak Chebyshev X-Greedy Algorithm. This greedy algorithm adds a step (Step 2 below) to the Weak X-Greedy Algorithm defined in [18].

Let $\mathcal{D}$ be a **complex dictionary** for a complex Banach space X—that is, $\|g\| = 1$ for each $g \in \mathcal{D}$ and X is the closed linear span over $\mathbb{C}$ of $\mathcal{D}$. Fix $0 < \tau < 1$ and let $\{\epsilon_n\}$ be nonnegative sequence of real numbers with $\epsilon_n \to 0$. Let $f \in X$, and define $f_0 := f$. We generate an approximation for f inductively for $m \geq 1$:

(1) Choose $\phi_m \in \mathcal{D}$ such that

$$\|f_{m-1}\| - \inf_{\lambda \in \mathbb{C}} \|f_{m-1} - \lambda\phi_m\| \geq \tau \Big[\|f_{m-1}\| - \inf_{\substack{\lambda \in \mathbb{C} \\ \phi \in \mathcal{D}}} \|f_{m-1} - \lambda\phi\| \Big].$$

(2) Choose $B_m \in \Phi_m = \mathrm{Span}_{\mathbb{C}}\{\phi_j\}_{j=1}^m$ to be the best approximant to f from Φ_m.

(3) Define $G_m \in \Phi_m$ to be any function satisfying

$$\|f - G_m\| \leq (1 + \epsilon_m)\|f - B_m\|,$$

so G_m is almost the best approximant to f from Φ_m.

(4) Define $f_m := f - G_m$.

Remark: Step 3 is not particularly important and can be omitted by setting $G_m = B_m$. However, the inclusion of Step 3 allows some flexibility in the choice of G_m.

The WCXGA and other greedy algorithms. The WCXGA seems to be particularly well-suited to Bergman spaces (at least in theory) as opposed to other greedy algorithms which which the WCXGA shares several features. Without giving precise definitions, let us briefly compare some of these other algorithms. By the way, the presence of the parameter τ in Step 1 is responsible for the term "weak" which appears in the names in these algorithms.

In the Weak X-Greedy Algorithm (WXGA), defined in [18], Step 2 (the "Chebyshev step") is omitted. We simply define $f_m := f_{m-1} - \lambda_m\phi_m$, where $\lambda := \lambda_m$ satisfies the inequality in Step 1 which is used to define ϕ_m. When X is a Hilbert space, the WXGA reduces to a weak version of the original "greedy algorithm" [9, 11] and is known to converge for every dictionary $\mathcal{D}$ [13]. However, there are no general convergence results for the WXGA in the Banach space setting.

The Weak Dual Greedy Algorithm (WDGA) omits the Chebyshev step and chooses ϕ_m by replacing Step 1 by a (weak) maximization of $F_{f_{m-1}}(\phi)$, where $F_{f_{m-1}} \in X^*$ is a norming functional for f_{m-1}. Recently, Ganichev and Kalton [6] proved that the WDGA converges for every dictionary $\mathcal{D}$ when X is an L_p space (or even a subspace of a quotient of an L_p space) for $1 < p < \infty$. However, there are no satisfactory convergence results for the WDGA in the setting of general Banach spaces.

The Weak Chebyshev Dual Greedy Algorithm (WCGA) is obtained from the WDGA by adding the "Chebyshev step" [17]. It was proved in [4] that the WCGA converges for every dictionary $\mathcal{D}$ when X is a reflexive Banach space with a **Fréchet differentiable** norm. In [4] the WCGA was also applied to Hardy space and was shown to converge for dictionaries which are "nice" (see below) and **uniformly integrable**. In particular, the holomorphic monomials form such a dictionary for $H_1(\mathbb{T}^n)$, and the WCGA converges in these spaces. Unfortunately, even for the Bergman spaces on the unit disk, the normalized holomorphic monomial dictionary $\{z^n\}$ is not uniformly integrable, so the proof in [4] of the convergence of the WCGA would break down in Bergman spaces. We do not see a way around this difficulty.

2. Convergence of the WCXGA

Theorem 2.1 below is the main theorem of this section; it says that the WCXGA converges for nice dictionaries in Banach spaces which have the weak-star Kadec-Klee property and are smooth. We start by explaining terminology.

A Banach space $X = Y^*$ has the **weak-star Kadec-Klee property** if whenever $\{f_n\} \subset X$ converges weak-star to f and $\|f\| = \lim_{n\to\infty} \|f_n\|$, then $\{f_n\}$ converges strongly to f—that is, $\{f_n\}$ converges in norm to f. We will study Kadec-Klee properties of Bergman spaces in particular in Section 3.1.

Smoothness is a notion which applies to real Banach spaces. Suppose X is a real Banach space. Define for $f, g \in X$ and for $u \in \mathbb{R}$

$$\rho_{f,g}(u) = \frac{\|f + ug\| + \|f - ug\| - 2\|f\|}{2}.$$

The point $f \in X$ is a **point of Gâteaux smoothness** if

$$\lim_{u\downarrow 0} \frac{\rho_{f,g}(u)}{u} = 0 \text{ for all } g \in X.$$

We say that $F_f \in X^*$ is a norming functional for f when $\|F_f\|_{X^*} = 1$ and $F_f(f) = \|f\|$; by the Hahn-Banach theorem, each $f \in X$ has at least one norming functional. It is known that f is a point of Gâteaux smoothness if and only if its associated norming functional F_f is unique. We say that a real Banach space X is **smooth** (or X **has a Gâteaux differentiable norm**) when every nonzero point in X is a point of Gâteaux smoothness. See [2] for more information on smoothness.

The assumption that every nonzero point of X is a point of Gâteaux smoothness guarantees that for *any* nonzero $f \in X$ and *any* dictionary $\mathcal{D}$,

$$\inf_{\substack{c\in\mathbb{R}\\ g\in\mathcal{D}}} \|f - cg\| < \|f\|.$$

If, like the Bergman spaces, X is a complex Banach space, we can apply the concept of smoothness to X by restricting multiplication to real scalars.

Lemma 2.1. *Let X be a smooth complex Banach space and let $\mathcal{D}$ be a dictionary for X. Then, for every nonzero $f \in X$,*

$$\inf_{\substack{\lambda\in\mathbb{C}\\ g\in\mathcal{D}}} \|f - \lambda g\| < \|f\|.$$

Proof: If we regard X as a real Banach space then $\tilde{\mathcal{D}} = \{g, ig : g \in \mathcal{D}\}$ is a (real) dictionary for X. Thus,

$$\inf_{\substack{t\in\mathbb{R}\\ g\in\mathcal{D}}} \|f - tg\| < \|f\|,$$

which gives us the result. □

If we knew instead that for every nonzero $f \in X$, there exist $g \in \mathcal{D}$ and $\lambda \in \mathbb{C}$ such that $\|f - \lambda g\| < \|f\|$, we could drop the requirement for smoothness. However, we will prove in Section 3.3 that the Bergman spaces are smooth Banach spaces.

Recall that a set $\mathcal{D} \subset X$ is a dictionary for X when $\|g\| = 1$ for each $g \in \mathcal{D}$ and X is the norm-closed linear span over $\mathbb{C}$ of $\mathcal{D}$. When X is a dual space, then a dictionary $\mathcal{D}$ is called **nice** if the norm-closed linear span of $\mathcal{E}$, denoted $[\mathcal{E}]$, is also weak-star closed for all $\mathcal{E} \subset \mathcal{D}$.

Theorem 2.1. *Suppose $X = Y^*$ has the weak-star Kadec-Klee property and is smooth as a real Banach space. Then the WCXGA converges for every nice dictionary $\mathcal{D}$ of X.*

Proof: We have

$$\|f_m\| \le (1 + \epsilon_m) \min_{1\le k\le m} \|f_k\|$$

since

$$\begin{aligned}\|f_m\| &\le (1+\epsilon_m)\|f-B_m\| \\ &= (1+\epsilon_m)\inf_{\phi\in\Phi_m}\|f-\phi\| \\ &\le (1+\epsilon_m)\min_{1\le k\le m}\|f-G_k\| \\ &= (1+\epsilon_m)\min_{1\le k\le m}\|f_k\|.\end{aligned}$$

Therefore, $\|f_m\|$ converges to some $\alpha \ge 0$. If $\alpha = 0$, then the WCXGA converges and we are done.

So, suppose $\alpha > 0$. Let $\mathcal{E} = \{\phi_n : n \ge 1\} \subset \mathcal{D}$. Now, by weak-star compactness, there is a subsequence $\{G_{n_k}\}$ such that $G_{n_k} \to x$ weak-star for some $x \in X$. Since $\mathcal{D}$ is nice, $\mathcal{E}$ is weak-star closed, and $x \in [\mathcal{E}]$. Now, $\operatorname{dist}(f, [\mathcal{E}]) = \alpha$. Therefore, $\|f - x\| \ge \operatorname{dist}(f, [\mathcal{E}]) = \alpha$. But

$$\alpha \le \|f-x\| \le \liminf_{k\to\infty}\|f-G_{n_k}\| = \lim_{k\to\infty}\|f_{n_k}\| = \alpha,$$

where the second inequality is due to weak-star lower semi-continuity of the norm of X. Therefore $\|f-x\| = \lim_{k\to\infty}\|f-G_{n_k}\|$; since X has the weak-star Kadec-Klee property, $f - G_{n_k} \to f - x$ strongly.

Since X is smooth as a real Banach space, there exist $\phi_0 \in \mathcal{D}$ and $\lambda_0 \in \mathbb{C}$ such that

$$\|f - x - \lambda_0\phi_0\| = \beta < \alpha = \|f-x\|.$$

Hence, for all sufficiently large k,

$$\begin{aligned}\|f - G_{n_k} - \lambda_0\phi_0\| &= \|f - x + x - G_{n_k} - \lambda_0\phi_0\| \\ &\le \|f - x - \lambda_0\phi_0\| + \|G_{n_k} - x\| < \frac{\alpha+\beta}{2}\end{aligned}$$

since $\|G_{n_k} - x\| \to 0$. In other words, there exist $\lambda_0 \in \mathbb{C}$, $\phi_0 \in \mathcal{D}$ with

$$\|f_{n_k} - \lambda_0\phi_0\| < \frac{\alpha+\beta}{2} \tag{1}$$

for k sufficiently large.

From the definition of the WCXGA,

$$\inf_{\lambda\in\mathbb{C}}\|f_{n_k} - \lambda\phi_{n_k+1}\| \le (1-\tau)\|f_{n_k}\| + \tau\inf_{\lambda\in\mathbb{C},\phi\in\mathcal{D}}\|f_{n_k} - \lambda\phi\|.$$

But Inequality (1) implies that there exists $\lambda \in \mathbb{C}$ such that

$$\|f_{n_k} - \lambda\phi_{n_k+1}\| \le (1-\tau)\|f_{n_k}\| + \tau\Big(\frac{\alpha+\beta}{2}\Big).$$

Therefore

$$\|f_{n_k+1}\| \le (1+\epsilon_{n_k})\Big[(1-\tau)\|f_{n_k}\| + \tau\Big(\frac{\alpha+\beta}{2}\Big)\Big].$$

Finally,

$$\begin{aligned}\alpha &= \lim_{k\to\infty} \|f_{n_k+1}\| \\ &\le (1-\tau)\lim_{k\to\infty}\|f_{n_k}\| + \tau\Big(\frac{\alpha+\beta}{2}\Big) \\ &= (1-\tau)\alpha + \tau\Big(\frac{\alpha+\beta}{2}\Big) \\ &= \alpha - \tau\Big(\frac{\alpha-\beta}{2}\Big) < \alpha,\end{aligned}$$

which is a contradiction. Therefore, α must be 0, and the WXCGA converges. □

We can state the following corollaries of Theorem 2.1.

Corollary 2.2. *The WCXGA converges for the Hardy space $H_1(\mathbb{T}^n)$ and the dictionary $\mathcal{D} = \{z^\alpha\}$.*

Corollary 2.3. *Let X be a smooth reflexive (real or complex) Banach space with the weak-star Kadec-Klee property. Then the WCXGA converges for every dictionary $\mathcal{D}$.*

Proof: Since X is reflexive, every norm-closed subspace of X is weakly closed. Therefore, every dictionary $\mathcal{D}$ is nice. □

3. Banach space properties of the Bergman spaces

There is a short survey of some Banach space properties of Bergman spaces in [20]; other good sources for general information about Bergman spaces are [1], [8], and [21].

Throughout this section, U is a bounded, open set in $\mathbb{C}^n$ and $X = L^1(U) \cap \mathcal{H}(U)$ denotes the Bergman space on U.

3.1. **Kadec-Klee properties.** We will show that X has the uniform Kadec-Klee (UKK) property for the topology of pointwise convergence. In order to do this, we will show that the unit ball of X is a normal family of functions. We will need the following theorem in what follows:

Theorem 3.1. (Montel's Theorem) *Let $\{f_\alpha\}$ be a family of holomorphic functions on U such that for every compact set $K \subset U$, there is a constant C_K such that $|f_\alpha(z)| < C_K$ for each α and for each $z \in K$. Then there is a sequence $\{f_n\} \subset \{f_\alpha\}$ which converges uniformly on compact sets of U to a holomorphic function f.*

The type of convergence in Montel's Theorem—uniform convergence on compact sets—is called **normal convergence**. The family $\{f_\alpha\}$ is called a **normal family** when every sequence in $\{f_\alpha\}$ has a subsequence which converges normally on U [7, p. 195].

Proposition 3.1. *The unit ball* $B(X) = \{f \in X : \|f\| = 1\}$ *is a normal family.*

Proof: For each $z \in U$, the point evaluation $P_z(f) = f(z)$ can be represented by integrating f against a function $g_z \in C_0(\overline{U})$—the set of continuous functions on U vanishing at the boundary of $\overline{U}$.

To see this, fix $z_0 \in U$. Choose r depending on z_0 such that the disk $\{|z - z_0| < r\} \subset U$, and define $h_r : \mathbb{R} \to \mathbb{R}$ by

$$h_r(x) = \begin{cases} \frac{r-x}{x}, & 0 \le x \le r \\ 0, & x > r. \end{cases}$$

The function $h_r(|w-z_0|)$ belongs to $C_0(\overline{U})$. Since f is holomorphic, f satisfies the Mean Value Equality. We use polar coordinates in $\mathbb{C}^n$ (see Fact 2 in Section 4) with the normalization $\lambda(B) = 1$ on the Euclidean unit ball B in $\mathbb{C}^n$. The measure σ is normalized surface measure on the Euclidean unit sphere S in $\mathbb{C}^n$. With these coordinates, we get

$$\begin{aligned} \int_U f(w) g_{z_0}(w) d\lambda(w) &= \int_{|w-z_0|<r} f(w) h_r(|z_0 - w|) d\lambda(w) \\ &= 2n \int_{t=0}^{r} t^{2n-1} h_r(t) \int_S f(z_0 - t\zeta) d\sigma(\zeta) dt \\ &= 2n f(z_0) \int_{t=0}^{r} t^{2n-1} h_r(t) dt = C_n(r) f(z_0) \end{aligned}$$

where $C_n(r)$ depends only on n and r. Therefore, point evaluation at $z_0 \in U$ is represented by $g_{z_0} = \frac{1}{C_n(r)} h_r(|w - z_0|)$. Clearly, $\|g_{z_0}\|_\infty \le \frac{1}{C_n(r)}$.

Now, let $K \subset U$ be compact; choose $r > 0$ such that for each $z_0 \in K$, the disk $\{z : |z - z_0| < r\} \subset U$. Let $f \in B(X)$. Now, $|f(z_0)| \le \frac{1}{C_n(r)}$ for all $z_0 \in K$, which means that $B(X)$ is a normal family. □

We now turn to Kadec-Klee properties of X. We say that X has the **uniform Kadec-Klee property (UKK) with respect to the topology of pointwise convergence** provided that for every $\epsilon > 0$, there exists $\delta > 0$ such that for all sequences $\{f_n\} \subset B(X)$ such that $f_n \to f$ pointwise and $\|f_n - f_m\| \ge \epsilon$ for $m \ne n$, then $\|f\| \le 1 - \delta$.

Theorem 3.2. *The Bergman space* X *has the UKK with respect to the topology of pointwise convergence.*

Suppose that $\{f_n\} \subset B(X)$, $f_n(z) \to f(z)$ for every $z \in U$, and $\|f_n - f_m\| \ge \epsilon$ for $m \ne n$. We will show that $\delta = \epsilon/2$ works in the definition of UKK with respect to the topology of pointwise convergence above.

Montel's Theorem guarantees that $f \in B(X)$. Let $\alpha > 0$ be arbitrary. Choose a compact set $K \subset U$ such that

$$\int_K |f(z)| d\lambda > \int_U |f(z)| d\lambda - \alpha.$$

Since $f_n(z) \to f(z)$ uniformly on K, we have

$$\limsup_{m,n\to\infty} \int_K |f_n(z) - f_m(z)| d\lambda = 0.$$

By hypothesis, $\int_U |f_m(z) - f_n(z)| d\lambda \geq \epsilon$ for $m \neq n$. So,

$$\liminf_{\substack{m\neq n \\ m,n\to\infty}} \int_{U\setminus K} |f_m(z) - f_n(z)| d\lambda \geq \epsilon.$$

Now we have

$$\limsup_{n\to\infty} \int_{U\setminus K} |f_n(z)| d\lambda \geq \frac{\epsilon}{2}.$$

But $\{f_n\} \subset B(X)$, so

$$\begin{aligned}
\frac{\epsilon}{2} &\leq \limsup_{n\to\infty} \int_{U\setminus K} |f_n(z)| d\lambda \\
&= \limsup_{n\to\infty} \left[\int_U |f_n(z)| d\lambda - \int_K |f_n(z)| d\lambda \right] \\
&\leq \limsup_{n\to\infty} \int_U |f_n(z)| d\lambda - \liminf_{n\to\infty} \int_K |f_n(z)| d\lambda \\
&\leq 1 - \liminf_{n\to\infty} \int_K |f_n(z)| d\lambda.
\end{aligned}$$

Finally,

$$\liminf_{n\to\infty} \int_K |f_n(z)| d\lambda \leq 1 - \frac{\epsilon}{2};$$

since $\{f_n\}$ converges uniformly to f on K, we have

$$\int_K |f(z)| d\lambda \leq 1 - \frac{\epsilon}{2}.$$

Therefore, $\|f\| \leq 1 + \alpha - \frac{\epsilon}{2}$. Since $\alpha > 0$ is arbitrary, $\|f\| \leq 1 - \frac{\epsilon}{2}$. $\square$

The hypothesis of Theorem 2.1 requires that X have the weak-star Kadec-Klee property; we will now show that if X has the UKK with respect to the topology of pointwise convergence, then X has the weak-star Kadec-Klee property. Recall that a Banach space $X = Y^*$ has the weak-star Kadec-Klee property if whenever $\{f_n\} \subset X$ converges weak-star to f and $\|f\| = \lim_{n\to\infty} \|f_n\|$, then $\{f_n\}$ converges strongly to f—that is, $\{f_n\}$ converges in norm to f. We will prove in Theorem 3.3 that

X is isometrically isomorphic to a dual space Y^*, where elements of Y are represented by continuous functions vanishing at the boundary of U. Further, Y contains point evaluations by Proposition 3.1. Therefore if $f_n \to f$ weak-star in X, then $f_n \to f$ pointwise as well.

The notion of uniformly Kadec-Klee can apply to other topologies besides the topology of pointwise convergence on X. In particular, we say that $X = Y^*$ has **the UKK with respect to the weak-star topology** if for every $\epsilon > 0$, there is a $\delta > 0$ such that for all sequences $\{f_n\} \subset B(X)$ such that $f_n \to f$ weak-star and $\|f_n - f_m\| > \epsilon$ for all $m \neq n$, then $\|f\| \leq 1 - \delta$. An equivalent definition is the following: $X = Y^*$ has the UKK with respect to the weak-star topology if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $\{f_n\}$ converges weak-star to f and $\|f_n\| \leq 1$ for all n and $\|f - f_n\| > \epsilon$ for all n, then $\|f\| < 1 - \delta$.

Now, the second definition for UKK with respect to the weak-star topology makes it easy to see that if X has the UKK for the topology of pointwise convergence, then X has the UKK for the weak-star topology since weak-star convergence implies pointwise convergence, which in turn implies that X is weak-star Kadec-Klee.

Let C be a closed, bounded, convex subset of a Banach space Y. A mapping $T : C \to C$ is **non-expansive** if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set C is said to have the **Fixed Point Property** if every non-expansive mapping $T : C \to C$ has a fixed point.

Corollary 3.2. *Let C be a weak-star compact, convex subset of the Bergman space X. Then C has the Fixed Point Property.*

Proof: Since X has the weak-star UKK property, it follows from a theorem of van Dulst and Sims [19] that every weak-star compact, convex C has the Fixed Point Property. □

3.2. **A predual of X.** The convergence of the WCXGA requires X to have the weak-star Kadec-Klee property. Not only do we identify X with a dual of a Banach space (that is, X has a predual), but we also show that X is isometrically isomorphic to that dual. In fact, X is isometrically isomorphic to the dual space of a quotient of $C_0(\overline{U})$, the space of continuous functions on U which vanish at the boundary of U, by Theorem 3.3 below. There are other ways to view X as a dual space; for example, in [21, Theorem 5.2.8], it is shown that the Bergman space on the disk in $\mathbb{C}$ can be identified with the dual of the little Bloch space with equivalence of norms. Here, we are concerned with showing that X is isometrically isomorphic to a dual space because the weak-star Kadec-Klee property is an isometric property.

Lemma 3.3. *The space $C_0(\overline{U})$ is norming for X—in other words,*

$$\|f\| = \sup\{\langle f, g\rangle : g \in C_0(\overline{U}), \|g\|_\infty \le 1\}.$$

Proof: Fix $f \in X$ with $\|f\| = 1$; f cannot be identically zero. Therefore, the set $Z(f) = \{z \in U : f(z) = 0\}$ has Lebesgue measure zero. In $\mathbb{C}$, this follows from the principle known as the Principle of Isolated Zeros. In order to extend this principle to $\mathbb{C}^n$ for $n > 1$, use induction.

Fix $\epsilon > 0$. Choose a compact set $K \subset U$ such that

$$\int_{U\setminus K} |f(z)| d\lambda < \frac{\epsilon}{2} \text{ and } |f(z)| \ge \delta > 0 \text{ for all } z \in K.$$

Define the function g_0 on K and $\partial\overline{U}$ so that

$$g_0(z) = \begin{cases} \frac{\overline{f(z)}}{|f(z)|} & z \in K \\ 0 & z \in \partial\overline{U}. \end{cases}$$

The function g_0 is continuous on $K \cup \partial\overline{U}$ and $|g_0(z)| \le 1$ for each $z \in K \cup \partial\overline{U}$. The Tietze Extension Theorem guarantees that g_0 has a continuous extension $g \in C_0(\overline{U})$ with $\|g\|_\infty \le 1$.

Now,

$$\begin{aligned} \int_U f(z)g(z)\, d\lambda &\ge \int_K f(z)g_0(z)\, d\lambda - \int_{U\setminus K} |f(z)| d\lambda \\ &= \int_K |f(z)| d\lambda - \int_{U\setminus K} |f(z)| d\lambda \\ &\ge 1 - \frac{\epsilon}{2} - \frac{\epsilon}{2} = 1 - \epsilon. \end{aligned}$$

Therefore, since ϵ was arbitrary, $\|f\| = \sup\{\langle f, g\rangle : g \in C_0(\overline{U}), \|g\|_\infty \le 1\}$. □

Theorem 3.3. *The Bergman space $X = L^1(U) \cap \mathcal{H}(U)$ is isometrically isomorphic to the dual space of $C_0(\overline{U})/(C_0(\overline{U}) \cap X^0)$, where $X^0 = \{g \in L^\infty : \langle f, g\rangle = 0$ for all $f \in X\}$.*

Proof: Let $Y = C_0(\overline{U})$. The natural inclusion $X \to Y^*$ is an isometry by Proposition 3.3. We show that X is a weak-star closed subspace of Y^*. By the Krein-Smulian Theorem it suffices to show that $B(X)$ is weak-star closed in Y^*.

Suppose $\{f_n\} \subset B(X)$. By Montel's Theorem, $\{f_n\}$ has a subsequence $\{f_{n_k}\}_{k=1}^\infty$ that converges uniformly on compact sets to some $f \in B(X)$. This implies that $\langle f_{n_k}, g\rangle \to \langle f, g\rangle$ for all $g \in C_0(\overline{U})$—that is, $f_{n_k} \to f$ weak-star in Y^*. Therefore $B(X)$ is weak-star closed.

By standard duality, there is a natural isometric isomorphism between X and the dual of the space $C_0(\overline{U})/(C_0(\overline{U}) \cap X^0)$. □

3.3. **Smoothness.** Up to now, we have considered the Bergman spaces as complex Banach spaces. However, if we view the Bergman spaces as real Banach spaces by restricting to multiplication by real scalars, we get the following result.

Theorem 3.4. *The Bergman space X is smooth when X is viewed as a real Banach space.*

Proof: We will show that if $f \in X$, $f \neq 0$, then f has a unique norming functional $F_f \in X^*$ with

$$F_f(f) = \|F_f\|_{X^*}\|f\| = \|f\|.$$

Let F_f be represented by $g \in L^\infty(U, \lambda)$, where g is a Hahn-Banach norming functional for f with $\|F_f\| = 1$. Then $\|g\|_\infty \leq 1$ and

$$F_f(f) = \int_U f(z)g(z)\, d\lambda \leq \int_U |f(z)|\, d\lambda = \|f\| = F_f(f).$$

Hence $g = \frac{\overline{f(z)}}{|f(z)|}$ on $U\backslash Z(f)$ a.e.; since $\lambda(U\backslash Z(f)) = 0$, $g = \frac{\overline{f(z)}}{|f(z)|}$ a.e. In particular, g is unique. If $h \in X$, then the unique real norming functional is given by

$$\mathrm{Re}[F_f(h)] = \mathrm{Re} \int_U h(z) \frac{\overline{f(z)}}{|f(z)|} d\lambda.$$

□

4. Nice dictionaries for Bergman spaces

Theorem 2.1 says that the WCXGA expansion of f with respect to a nice dictionary converges to f in norm. A dictionary $\mathcal{D}$ is nice whenever the norm-closed linear span of $\mathcal{E}$, denoted $[\mathcal{E}]$, is also weak-star closed for all $\mathcal{E} \subset \mathcal{D}$. We begin by asking whether Bergman spaces have nice dictionaries, and then we show that the holomorphic monomial dictionaries for the Bergman spaces on the unit ball and unit polydisk in $\mathbb{C}^n$ are nice dictionaries.

4.1. **Existence of nice dictionaries.** Recall that the sequence $\{x_n\} \subset X$ is a **Schauder basis** for X if for each $x \in X$ there exists a unique sequence of scalars $\{c_n\}$ such that $\|x - \sum_{k=1}^n c_k x_k\| \to 0$ as $n \to \infty$.

Proposition 4.1. *Suppose that Y is a Banach space and that Y^* has a Schauder basis. Then Y^* has a nice Schauder basis.*

Proof: By a theorem of Johnson and Rosenthal [10], Y^* has a Schauder basis $\{e_n\}$ such that the dual functionals $\{e_n^*\}$ are weak-star continuous (i.e. $e_n^* \in Y$).

Let $\mathcal{D} = \{e_n\}$ and let $\mathcal{E} \subset \mathcal{D}$. Then

$$[\mathcal{E}] = \{x \in Y^* : x = \sum_{n \in \mathcal{E}} e_n^*(x) e_n\} = \bigcap_{n \notin \mathcal{E}} \{x \in Y^* : e_n^*(x) = 0\}$$

which is weak-star closed. □

Corollary 4.2. *Let X be the unweighted Bergman space on the domain U. Suppose that there exists a bounded linear projection from $L^1(U, \lambda)$ onto X. Then X is isomorphic to ℓ_1. Consequently, X has a nice basis.*

Proposition 4.3. *Suppose that there is a bounded linear projection from $L^1(U, \lambda)$ onto X. Then X is isomorphic as a Banach space to ℓ^1.*

Proof: X is a separable dual space by Theorem 3.3, therefore X has the Radon-Nikodym property (see, for example, [3]). By a theorem of Lewis and Stegall [12], all complemented subspaces of $L^1(\lambda)$ with the Radon-Nikodym property are isomorphic to ℓ^1. The existence of a nice basis now follows from Proposition 4.1. □

Forelli and Rudin have shown that when $X = L^1(U, \lambda) \cap \mathcal{H}(U)$, where U is the unit ball in $\mathbb{C}^n$, there are many bounded linear projections from $L^1(U, \lambda)$ onto X [5]. Stoll has shown that there are bounded linear projections from $L^1(U, \lambda)$ onto $L^1(U, \lambda) \cap \mathcal{H}(U)$ when U is a bounded symmetric domain in $\mathbb{C}^n$[15]. The polydisk in $\mathbb{C}^n$ is an example of a bounded symmetric domain.

In Sections 4.2 and 4.3, we show that the holomorphic monomials $\{z^\alpha\}$ form nice dictionaries for the complex Bergman spaces on the unit ball in $\mathbb{C}^n$ and the unit polydisk in $\mathbb{C}^n$. The following is standard notation which allows us to work with the monomials: if $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, then $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n}$. In addition, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\alpha! = (\alpha_1)! \ldots (\alpha_n)!$.

4.2. **Nice dictionaries for the Bergman space on the unit ball and polydisk.** Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, and let B denote the open unit ball $\{z : |z| < 1\} \subset \mathbb{C}^n$, where $|z| = \left(\sum_{i=1}^n |z_i|^2\right)^{\frac{1}{2}}$. We let ν denote normalized Lebesgue measure on B, so that $\nu(B) = 1$. In addition, we let $S = \{z : |z| = 1\}$ denote the surface of B with normalized Lebesgue measure σ, so that $\sigma(S) = 1$. The weak-star topology on X is induced by $C_0(\overline{B})$. We will use the following facts about integration on B, all of which come from the first chapter of [14].

Facts:

(1) Monomials are orthogonal to each other:

$$\int_S z^\alpha \overline{z}^\beta d\sigma = \begin{cases} 0 & \alpha \neq \beta \\ \frac{(n-1)!\alpha!}{n-1+\alpha!} & \alpha = \beta. \end{cases}$$

If we integrate over B instead, $n-1$ is replaced with n.

(2) If g is integrable on B, then

$$\int_B g(z)\, d\nu(z) = 2n \int_0^1 r^{2n-1} \left(\int_S g(r\zeta)\, d\sigma(\zeta) \right) dr.$$

We used these polar coordinates in Proposition 3.1.

(3) Every holomorphic function on B has a multi-power series $f(z) = \sum_\alpha c_\alpha z^\alpha$ valid for all $z \in B$. This power series converges absolutely and uniformly for all z with $|z| \leq r$ for fixed $r < 1$.

(4) If f is holomorphic on B, then $\int_S |f(r\zeta)|\, d\sigma(\zeta)$ is an increasing function of r for $0 < r < 1$.

Theorem 4.1. *The dictionary $\{z^\alpha\}$ is a nice dictionary for X. For any set of multi-indices A, the norm-closed linear span of $\{z^\alpha : \alpha \in A\}$ is the same as the weak-star closed linear span of $\{z^\alpha : \alpha \in A\}$.*

Proof: We will prove that $\{z^\alpha\}$ is nice in two steps: first, we show that the weak-star closure of $\mathrm{Span}\{z^\alpha : \alpha \in A\}$ consists of power series in z^α where $\alpha \in A$, and then we will show that power series in z^α where $\alpha \in A$ is norm-closed.

Suppose A is a collection of multi-indices, and suppose $f(z) = \sum c_\gamma z^\gamma \in X$ lies in the weak-star closure of $\mathrm{Span}\{z^\alpha : \alpha \in A\}$. We show that $c_\gamma = 0$ unless $\gamma \in A$.

Consider $y = h(x) = \max\{1 - 2x, 0\}$, and fix a multi-index $\beta \notin A$. Note that $h(|z|)\overline{z}^\beta \in C_0(\overline{B})$. We use the convergence from Fact 3 to interchange summation and integration and apply polar coordinates from Fact 2:

$$\begin{aligned}
\int_B f(z)\Big(h(|z|)\, \overline{z}^\beta\Big)\, d\nu(z) &= \sum \int_B c_\gamma h(|z|) z^\gamma \overline{z}^\beta d\nu(z) \\
&= \sum \int_B c_\gamma h(|r\zeta|)(r\zeta)^\gamma (\overline{r\zeta})^\beta d\nu(r\zeta) \\
&= \sum 2n \int_0^1 r^{2n-1} h(|r|) \left(\int_S (r\zeta)^\gamma (\overline{r\zeta})^\beta\, d\sigma(\zeta) \right) dr.
\end{aligned}$$

Fact 1 tells us that the inner integral is 0 unless $\gamma = \beta$; when $\gamma = \beta$,

$$\int_B f(z)\ h(|z|)\ \overline{z}^\beta\ d\nu(z) = k_\beta c_\beta$$

where $k_\beta > 0$ depends only on β. Thus, for all $f \in \text{Span}\{z^\alpha : \alpha \in A\}$,

$$\int_B f(z)\ h(|z|)\ \overline{z}^\beta\ d\nu(z) = 0.$$

Now, for all f belonging to the *weak-star* closure of $\text{Span}\{z^\alpha : \alpha \in A\}$,

$$\int_B f(z)\ h(|z|)\ \overline{z}^\beta\ d\nu(z) = k_\beta c_\beta = 0$$

by weak-star continuity. Since $k_\beta > 0$, $c_\beta = 0$ for all $\beta \notin A$.

Now we show that if $f(z) = \sum_{\alpha\in A} c_\alpha z^\alpha \in X$, then f belongs to the *norm* closure of $\text{Span}\{z^\alpha : \alpha \in A\}$—that is, for any $\epsilon > 0$ there is a finite sum $g(z) = \sum_{|\alpha|<N} c_\alpha z^\alpha$ such that $\|f - g\|_X < \epsilon$.

Fix $\epsilon > 0$. There exists $r_0 < 1$ such that

$$\int_{r_0<|z|<1} |f(z)|\ d\nu(z) < \frac{\epsilon}{4}.$$

The power series $\sum_\alpha c_\alpha z^\alpha$ converges uniformly absolutely on $|z| \le r_0$. Hence there exists $s_0 < 1$ such that $|f(z) - f(s_0 z)| < \frac{\epsilon}{4}$ for all $|z| < r_0$. By Fact 4, we have

$$\int_{r_0<|z|<1} |f(s_0 z)|\ d\nu(z) \le \int_{r_0<|z|<1} |f(z)|\ d\nu(z) < \frac{\epsilon}{4},$$

and therefore we also have

$$\|f(z)-f(s_0 z)\|_X = \int_{|z|\le r_0} |f(z)-f(s_0 z)|\ d\nu(z) + \int_{r_0<|z|<1} |f(z)-f(s_0 z)|\ d\nu(z) < \frac{3\epsilon}{4}.$$

But $f(s_0 z) = \sum_{\alpha\in A} c_\alpha (s_0 z)^\alpha$, and this power series converges uniformly absolutely on B since $s_0 < 1$. So there exists N such that

$$\left| f(s_0 z) - \sum_{|\alpha|\le N} c_\alpha s_0^{|\alpha|} z^\alpha \right| < \frac{\epsilon}{4}$$

for all $z \in B$. Thus

$$\left\| f(s_0 z) - \sum_{|\alpha|\le N} c_\alpha s_0^{|\alpha|} z^\alpha \right\| < \frac{\epsilon}{4}.$$

Finally,

$$\left\|f(z) - \sum_{|\alpha|\le N} c_\alpha s_0^{|\alpha|} z^\alpha\right\| \le \|f(z) - f(s_0 z)\| + \left\|f(s_0 z) - \sum_{|\alpha|\le N} c_\alpha s_0^{|\alpha|} z^\alpha\right\|$$

$$\le \frac{3\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.$$

But $\sum_{\alpha\in A} c_\alpha (s_0 z)^\alpha$ is in $\mathrm{Span}\{z^\alpha : \alpha \in A\}$. □

Corollary 4.4. *For $n \ge 1$, the WCXGA converges for the Bergman space on the unit ball in $\mathbb{C}^n$ with the dictionary $\{z^\alpha\}$.*

4.3. **A nice dictionary for the Bergman space on the polydisk.** Let U_n denote the polydisk $\{z \in \mathbb{C}^n : |z_i| < 1, 1 \le i \le n\}$, and let ν denote normalized Lebesgue measure on U_n, so that $\nu(U_n) = 1$. We consider the Bergman space X on the polydisk.

Theorem 4.2. *The set $\{z^\alpha\}$ is a nice dictionary for X.*

Proof: The proof is essentially the same as the proof of Theorem 4.1, except we use the products $\prod_{i=1}^n h(|z_i|)\overline{z}^\beta \in C_0(\overline{U_n})$ in order to show that if $f = \sum c_\alpha z^\alpha \in X$ lies in the weak-star closure of $\mathrm{Span}\{z^\alpha : \alpha \in A\}$, then $c_\alpha = 0$ unless $\alpha \in A$. □

Corollary 4.5. *For $n \ge 1$, the WCXGA converges for the Bergman space on the polydisk in $\mathbb{C}^n$ with the dictionary $\{z^\alpha\}$.*

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