# ON THE OPTIMALITY OF A THEOREM OF ELTON ON $\ell_1^n$ SUBSYSTEMS

S. J. DILWORTH AND DENKA KUTZAROVA

Abstract. We exhibit examples which show that a substantial strengthening of the hypothesis in the almost isometric part of a theorem of Elton on $\ell_1^n$ subsystems does not lead to a substantially stronger conclusion.

A well-known theorem of Elton [1, Theorem 1] on the existence of $\ell_1^n$ subsystems is in two parts. The second part, which is 'almost isometric' in character, may be formulated as follows.

**Theorem E.** *Let* $\alpha \in (0, 1/2)$ *and let* $\beta \in (0, 1)$*. There exists* $\delta < 1$ *(depending only on* $\alpha$ *and* $\beta$*) such that if* $(e_i)_{i=1}^n$ *are vectors in the unit ball of a real Banach space* $X$ *such that*

$$\operatorname*{average}_{\pm} \left\| \sum_{i=1}^n \pm e_i \right\| \geq \delta n, \tag{1}$$

*(the average taken over all choices of* $\pm$*) then there exists* $A \subseteq \{1, 2, \dots, n\}$*, with* $|A| \geq (1/2 - \alpha)n$*, such that*

$$\left\| \sum_{i \in A} a_i e_i \right\| \geq \beta \sum_{i \in A} |a_i|$$

*for all scalars* $(a_i)_{i \in A}$*.*

We are not concerned here with the first part of [1, Theorem 1], which is 'isomorphic' in character: the reader is referred to [1] for this result (and to [3] for the case of complex scalars); the reader is referred to [4] and [5] for further isomorphic results related to Theorem E.

An example due to Szarek [1, p. 121] shows that it is not possible to choose $A$ in Theorem E to satisfy $|A| \geq (1/2 + \alpha)n$. The theorem below answers

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a question raised in [1, p. 114] by showing that it is not possible to choose $A$ in Theorem E to satisfy $|A| \geq (1/2+\alpha)n$ even if the hypothesis (1) is replaced by the stronger hypothesis

$$\min_{\pm} \left\| \sum_{i=1}^{n} \pm e_i \right\| \geq \delta n.$$

In fact, given $\beta > 0$, our example can be constructed (see (ii) of the Theorem) to satisfy

$$\min_{\pm} \left\| \sum_{i=1}^{n} \pm e_i \right\| \geq n - \beta.$$

Recall that a sequence $(y_i)_{i=1}^n$ in a Banach space $X$ is *suppression* 1-*unconditional* if, whenever $A \subseteq B \subseteq \{1,2,\dots,n\}$, then $\|\sum_A a_i y_i\| \leq \|\sum_B a_i y_i\|$ for all scalars $(a_i)$. In the following, $(e_i)_{i=1}^n$ denotes the standard basis of $\mathbb{R}^n$ and $\|\cdot\|_1$ denotes the $\ell_1^n$ norm. For a vector $x = \sum_{i=1}^n x_i e_i$, $\operatorname{supp} x$ denotes the set $\{1 \leq i \leq n \colon x_i \neq 0\}$.

**Theorem.** *Let $\alpha \in (0,1/2)$ and let $\beta \in (0,1)$. For all sufficiently large $n$ there exists a norm $\|\cdot\|$ on $\mathbb{R}^n$ with the following properties:*

(i) *$(e_i)_{i=1}^n$ is a suppression 1-unconditional normalized basis of $(\mathbb{R}^n, \|\cdot\|)$.*

(ii)

$$\left\| \sum_{i=1}^{n} \pm e_i \right\| \geq n - \beta$$

*for all choices of signs.*

(iii) *For every $A \subseteq \{1,2,\dots,n\}$, with $|A| = 1 + \lceil (1/2+\alpha)n \rceil$, there exists a nonzero vector $x$, with $\operatorname{supp} x \subseteq A$, such that*

$$\|x\|_1 \geq (1+\eta(\alpha,\beta))\|x\|, \tag{2}$$

*where*

$$\eta(\alpha,\beta) = \frac{\alpha\beta}{(3\alpha+1)\lfloor (2\ln 2)/\alpha^2 \rfloor - \alpha\beta}.$$

*Remark.* Note that $\eta(\alpha,\beta) \geq c\alpha^3\beta$, where $c$ is an absolute constant. This *linear* dependence of $\eta$ on $\beta$ is optimal since (ii), $\|e_i\| \leq 1$, and the triangle inequality imply

$$\|x\| \geq (1-\beta)\|x\|_1 \qquad (x \in \mathbb{R}^n).$$

The following probabilistic lemma will be used to construct $\|\cdot\|$. (Here $A \triangle B$ denotes the *symmetric difference* of $A$ and $B$.)

**Lemma.** *Let $\alpha \in (0,1)$. For all sufficiently large $n$ there exist $n$ sets $S_i \subseteq \{1,2,\dots,n\}$ $(1 \le i \le n)$ satisfying the following: for every $S \subseteq \{1,2,\dots,n\}$, we have*

$$|\{1 \le i \le n\colon \min(|S \triangle S_i|, |(I \setminus S) \triangle S_i|) \le (1/2 - \alpha/2)n\}| \le \frac{2\ln 2}{\alpha^2}.$$

*Proof.* First we recall a well-known estimate (see e.g. [2] for a more general inequality). Let $(\varepsilon_m)_{m=1}^\infty$ be a sequence of independent Bernoulli random variables (defined on a probability space $(\Omega, \mathbb{P})$) taking the values 1 and $-1$ with probability 1/2. Then, for $\alpha > 0$ and $n \ge 1$, we have

$$\mathbb{P}\left(\sum_{m=1}^{n} \varepsilon_m \ge \alpha n\right) \le \exp(-n\alpha^2/2). \tag{3}$$

Set $k = \lfloor (2\ln 2)/\alpha^2 \rfloor + 1$. We shall choose the sets $S_i$ *independently* with the uniform distribution. Fix $S \subseteq \{1,2,\dots,n\}$. Then, for each $1 \le i \le n$, we have

$$\mathbb{P}(|S \triangle S_i(\omega)| \le (1/2 - \alpha/2)n) \le \exp(-n\alpha^2/2).$$

Indeed, this is precisely equivalent to (3) if we identify subsets of $\{1,2,\dots,n\}$ with sequences of 1's and $-1$'s in the obvious way. Hence

$$\mathbb{P}(\min(|S \triangle S_i(\omega)|, |(I \setminus S) \triangle S_i(\omega)|) \le (1/2 - \alpha/2)n) \le 2\exp(-n\alpha^2/2).$$

Now fix $1 \le j_1 < j_2 < \dots < j_k \le n$. By independence, we have

$$\begin{aligned}\mathbb{P}(\min(|S \triangle S_i(\omega)|, |(I \setminus S) \triangle S_i(\omega)|) &\le (1/2 - \alpha/2)n \text{ for all } i \in \{j_1,\dots,j_k\}) \\ &\le 2^k \exp(-kn\alpha^2/2).\end{aligned}$$

So the probability that *there exists* $S \subseteq \{1,2,\dots,n\}$ and that there exist indices $1 \le j_1 < j_2 < \dots < j_k \le n$ for which

$$\min(|S \triangle S_i(\omega)|, |(I \setminus S) \triangle S_i(\omega)|) \le (1/2 - \alpha/2)n$$

for all $i \in \{j_1,\dots,j_k\}$ is at most

$$2^n \binom{n}{k} 2^k \exp(-kn\alpha^2/2) = \binom{n}{k} 2^k \exp(-n(k\alpha^2/2 - \ln 2)).$$

Since $k\alpha^2/2 - \ln 2 > 0$, this probability is less than 1 for all sufficiently large $n$. So, for all sufficiently large $n$, there exists $\omega \in \Omega$ such that $S_i(\omega)$ $(1 \le i \le n)$ satisfy the conclusion. □

Now we start on the proof of the Theorem. Let $I = \{1,2,\dots,n\}$ and let $(S_i)_{i=1}^n$ satisfy the conclusion of the Lemma for $\alpha \in (0,1/2)$. Let $k(\alpha) = \lfloor (2\ln 2)/\alpha^2 \rfloor$ and let $\gamma = \beta/k(\alpha)$. Note that $\gamma \in (0,1)$.

For $1 \le i \le n$, we say that a set $S \subseteq I$ is *$i$-large* if either $|S \triangle S_i| \le (1/2 - \alpha/2)n$ or $|(I \setminus S) \triangle S_i| \le (1/2 - \alpha/2)n$. Note that, for each $1 \le i \le n$, the collection of all $i$-large sets is closed under complementation.

Let $y = (y_i)_{i \in I}$ be a vector whose coordinates belong to the interval $[-1, 1]$. We set $P(y) = \{i \in I \colon y_i > 1 - \gamma\}$ and $N(y) = \{i \in I \colon y_i < -1 + \gamma\}$. For $S \subseteq I$, we say that $y$ is $S$-*admissible* and that $y$ *is obtained from* $S$ if the following conditions hold:

(a) $|y_i| \le 1 - \gamma$ whenever $S$ is $i$-large.
(b) $P(y) \subseteq S$ and $N(y) \subseteq I \setminus S$.

Note that if $y$ is $S$-admissible then $-y$ is $(I \setminus S)$-admissible. This follows from the fact that the collection of $i$-large sets is closed under complementation.

A vector $y$ is said to be *admissible* if $y$ is $S$-admissible for some $S \subseteq I$. Let $F$ denote the collection of all admissible vectors. Then $F$ is symmetric, i.e. if $y \in F$ then $-y \in F$.

Now we can define the norm $\|\cdot\|$:

$$\left\| \sum_{i \in I} x_i e_i \right\| = \max_{y \in F} \sum_{i \in I} x_i y_i. \tag{4}$$

The symmetry of $F$ guarantees that (4) defines a norm. The fact that this norm is suppression 1-unconditional is an immediate consequence of the following easily checked property of $F$: if $y \in F$ and $z$ is obtained from $y$ by replacing some of the coordinates of $y$ by zeros, then $z \in F$. It is also easy to check that $\|e_i\| = 1$ for all $1 \le i \le n$.

*Proof of (ii).* Let $\eta = (\eta_i)_{i=1}^n$ be a choice of signs. Define $y = (y_i)$ thus:

$$y_i = \begin{cases} \eta_i & \text{if } P(\eta) \text{ is } \textit{not } i\text{-large}, \\ (1-\gamma)\eta_i & \text{if } P(\eta) \text{ is } i\text{-large}. \end{cases}$$

Clearly, $y$ is $P(\eta)$-admissible, so $y \in F$. By the Lemma, $P(\eta)$ is $i$-large for at most $k(\alpha)$ indices $i$. Thus

$$\left\| \sum_{i=1}^n \eta_i e_i \right\| \ge \sum_{i=1}^n \eta_i y_i \ge \sum_{i=1}^n \eta_i^2 - k(\alpha)\gamma = n - \beta.$$

□

*Proof of (iii).* Suppose $A \subset I$ with $|A| = 1 + \lceil (1/2 + \alpha)n \rceil$. Choose $i_0 \in A$ and set $\tilde{A} = A \setminus \{i_0\}$ (so that $|\tilde{A}| = \lceil (1/2 + \alpha)n \rceil$). We define a vector $x$, with $\operatorname{supp} x = A$, thus:

$$x_i = \begin{cases} |\tilde{A}| - (1/2 + \alpha/2)n & \text{for } i = i_0, \\ 1 & \text{for } i \in \tilde{A} \cap S_{i_0}, \\ -1 & \text{for } i \in \tilde{A} \cap (I \setminus S_{i_0}), \\ 0 & \text{otherwise.} \end{cases}$$

Now let us show that $\|x\|$ satisfies (2). Let $y$ be an admissible vector that is obtained from $S \subseteq I$. Suppose that

$$|\tilde{A} \cap S_{i_0} \cap P(y)| + |\tilde{A} \cap (I \setminus S_{i_0}) \cap N(y)| > (1/2 + \alpha/2)n. \tag{5}$$

Since $P(y) \subseteq S$ and $N(y) \subseteq I \setminus S$, we have

$$|S_{i_0} \cap S| + |(I \setminus S_{i_0}) \cap (I \setminus S)| > (1/2 + \alpha/2)n.$$

Thus,

$$|S_{i_0} \triangle S| < (1/2 - \alpha/2)n.$$

So $S$ is $i_0$-large. Thus $|y_{i_0}| \leq 1 - \gamma$. Hence

$$\begin{aligned} \sum_{i \in I} x_i y_i &= x_{i_0} y_{i_0} + \sum_{i \in \tilde{A}} x_i y_i \\ &\leq (1-\gamma)(|\tilde{A}| - (1/2 + \alpha/2)n) + |\tilde{A}|. \end{aligned} \tag{6}$$

Note that if $i \in \tilde{A} \setminus ((\tilde{A} \cap S_{i_0} \cap P(y)) \cup (\tilde{A} \cap (I \setminus S_{i_0}) \cap N(y)))$ then $x_i y_i \leq 1 - \gamma$. It follows that if (5) does not hold, then

$$\begin{aligned} \sum_{i \in I} x_i y_i &\leq |x_{i_0}| + |\tilde{A}| - \gamma(|\tilde{A}| - (1/2 + \alpha/2)n) \\ &= (|\tilde{A}| - (1/2 + \alpha/2)n) + |\tilde{A}| - \gamma(|\tilde{A}| - (1/2 + \alpha/2)n) \\ &= (1-\gamma)(|\tilde{A}| - (1/2 + \alpha/2)n) + |\tilde{A}|. \end{aligned} \tag{7}$$

It follows from (6) and (7) that

$$\|x\| = \sup_{y \in F} \sum_{i=1}^{n} x_i y_i \leq (1-\gamma)(|\tilde{A}| - (1/2 + \alpha/2)n) + |\tilde{A}|. \tag{8}$$

But

$$\begin{aligned} \|x\|_1 &= |x_{i_0}| + |\tilde{A}| \\ &= (|\tilde{A}| - (1/2 + \alpha/2)n) + |\tilde{A}| \\ &\geq \|x\| + \gamma(|\tilde{A}| - (1/2 + \alpha/2)n) \end{aligned}$$

(by (8))

$$\geq \left(1 + \frac{\gamma(|\tilde{A}| - (1/2 + \alpha/2)n)}{(1-\gamma)(|\tilde{A}| - (1/2 + \alpha/2)n) + |\tilde{A}|}\right) \|x\|$$

(by (8) again)

$$\geq \left(1 + \gamma \left\{(1-\gamma) + \frac{1 + 2\alpha}{\alpha}\right\}^{-1}\right) \|x\|$$

since $|\tilde{A}| \geq (1/2 + \alpha)n$. Substituting $k(\alpha) = \lfloor (2\ln 2)/\alpha^2 \rfloor$ and $\gamma = \beta/k(\alpha)$ into the above inequality yields (2). □

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Department of Mathematics, University of South Carolina, Columbia, SC 29208, U.S.A.
*E-mail address*: `dilworth@math.sc.edu`

Institute of Mathematics, Bulgarian Academy Of Sciences, Sofia, Bulgaria.
Current address: Department of Mathematics, University of South Carolina, Columbia, SC 29208, U.S.A.
*E-mail address*: `denka@math.sc.edu`