Isomorphisms and strictly singular operators in mixed Tsirelson spaces

Denka Kutzarova\textsuperscript{1,2}

Institute of Mathematics, Bulgarian Academy of Sciences, Sofia, Bulgaria

Antonis Manoussakis

Department of Sciences, Technical University of Crete, 73100 Chania (Crete), Greece

Anna Pelczar-Barwacz\textsuperscript{3,4}

Institute of Mathematics, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland

Abstract

We study the family of isomorphisms and strictly singular operators in mixed Tsirelson spaces and their modified versions setting. We show sequential minimality of modified mixed Tsirelson spaces \( T_M([S_n, \theta_n]) \) satisfying some regularity conditions and present results on existence of strictly singular non-compact operators on subspaces of mixed Tsirelson spaces defined by the families \((A_n)\) and \((S_n)\).

Keywords: quasiminimality; strictly singular operator; mixed Tsirelson space

Introduction

In the celebrated paper \[20\] W.T. Gowers started his classification program for Banach spaces. The goal is to identify classes of Banach spaces which are

1. hereditary, i.e. if a space belongs to a given class, then all of its closed infinite dimensional subspaces as well,
2. inevitable, i.e. any Banach space contains an infinite dimensional subspace in one of those classes,
3. defined in terms of family of bounded operators in the space.

The famous Gowers’ dichotomy brought first two classes: spaces with unconditional basis and hereditarily indecomposable spaces. The further classification, described in terms of isomorphisms, concerned minimality and strict quasiminimality. A Banach space \( X \) is \textit{minimal} if every closed infinite dimensional subspace of \( X \) contains a further subspace isomorphic to \( X \). A Banach space \( X \) is called \textit{quasiminimal} if any two infinite dimensional subspaces \( Y, Z \) of \( X \) contain further isomorphic subspaces. The classical spaces \( \ell_p, 1 \leq p < \infty \), \( c_0 \) are minimal and the Tsirelson space \( T[S_1, 1/2] \) is the first known strictly quasiminimal space (i.e. without minimal subspaces) \[15\]. The results of W.T. Gowers led to the question of the refinement of the classes and classification of already known Banach spaces. A further step in the first direction was made by the third named author \[31\], who proved that a strictly quasiminimal Banach space contains a subspace with

\textsuperscript{1}Present address: Department of Mathematics, University of Illinois at Urbana-Champaign, USA
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\textsuperscript{4}Corresponding author
no subsymmetric sequence. An extensive refinement of list of the classes and study of examples were made recently by V. Ferenczi and C. Rosendal [16, 17].

The mixed Tσiren space $T[(\mathcal{M}_n, \theta_n)_n]$, for $\mathcal{M}_n = \mathcal{A}_n$ or $\mathcal{S}_n$, as the basic examples of spaces not containing $\ell_p$ or $c_0$, form a natural class to be studied with respect to the classification program. The first step was made by T. Schlumprecht [5], who proved that his famous space $S = T[(\mathcal{A}_n, 1/\log_2(n + 1))_n]$ is complementably minimal. The result of Schlumprecht holds for a certain class of mixed Tσiren spaces $T[(\mathcal{A}_k, \theta_n)_n]$ by [27]. On the other hand, the Tafnir’s space $T[(\mathcal{A}_n, c/\sqrt{n})_n]$ [36] is not minimal by [21]. However the original Tσiren space $T[\mathcal{S}_n, 1/2]$ is not minimal [15], every normalized block sequence is equivalent to a subsequence of the basis. We show that mixed Tσiren spaces $T[(\mathcal{A}_n, \theta_n)_n]$, for which Tafnir space is a prototype, are saturated with subspaces with this "blocking principle".

V. Ferenczi and C. Rosendal [16] introduced and studied a stronger notion of quasiminimality. A Banach space $X$ with a basis is sequentially minimal [16], if any block subspace of $X$ contains a block sequence $(x_n)$ such that every block subspace of $X$ contains a copy of a subsequence of $(x_n)$. The related notions in mixed Tσiren spaces defined by families $(\mathcal{S}_n)$ and their relation to existence of $\ell_p$-spreading models were studied in [25, 22]. In [29] it was shown that the spaces $T[(\mathcal{A}_n, \theta_n)_n]$, as well as $T[(\mathcal{S}_n, \theta_n)_n]$ satisfying the regularity condition $\theta_n/\theta^n \to \infty$, where $\theta = \lim_n \theta_n/n$, are sequentially minimal. We show that the modified mixed Tσiren spaces $T_M[(\mathcal{S}_n, \theta_n)_n]$ with the above property are also sequentially minimal.

The major tool in the study of mixed Tσiren spaces $T[(\mathcal{S}_n, \theta_n)_n]$ are the tree-analysis of norming functionals and the special averages introduced in [7], see also [11]. The basic idea to prove quasiminimality is to produce in every subspace a sequence of appropriate special averages of rapidly increasing lengths and show these sequences span isomorphic subspaces. The major obstacle in study of modified mixed Tσiren spaces is estimating the norms of splitting of a vector into pairwise disjoint parts instead of consecutive parts as in non-modified setting. In order to overcome it, we introduced special types of averages, so-called Tσiren averages, describing in fact local representation of the Tσiren space $T[\mathcal{S}_n, \theta]$ with $\theta = \sup_n \theta_n/\theta^n$, in the considered space. Then we are able to estimate the action of a norming functional on a linear combination of Tσiren averages by the action of a norming functional on suitable averages in the Tσiren space $T[\mathcal{S}_n, \theta]$. Using those estimations we prove the sequential minimality of modified mixed Tσiren space satisfying the regularity condition. Special averages, a weaker form of Tσiren averages, are also the main tool for proving arbitrary distortability of $T_M[(\mathcal{S}_n, \theta_n)_n]$ in case $\theta_n/\theta^n \to 0$, the result known before in non-modified setting under the condition $\theta_n/\theta^n \to 0$ [3].

In the second part of the paper we deal with the existence of strictly singular non-compact operators in mixed Tσiren spaces. The existence of non-trivial strictly singular operators, i.e. operators whose norm restriction to an infinite dimensional subspace is an isomorphism, was also studied in context of classification program of Banach space, both in search for sufficient conditions and examples on known spaces. A space on which all the bounded operators are compact perturbations of multiple of the identity was constructed recently by S.A. Argyros and R. Haydon [10], who solved "scalar-plus-compact" problem. The existence of strictly singular non-compact operators was shown on Gowers-Maurey space and Schlumprecht space [6], as well as on a class of spaces defined by families $(\mathcal{S}_n)_n$ [19]. T. Schlumprecht [35], studying the richness of the family of operators on a Banach space in connection with the "scalar-plus-compact" problem, defined two classes of Banach spaces. Class 1 refers to a variation of a "blocking principle", while a space belongs to Class 2 if and only if it admits a strictly singular non-compact operator in any subspace (see Def. 3.3). T. Schlumprecht asked if any Banach space contains a subspace with a basis which is either of Class 1 or Class 2. We show that a mixed Tσiren space with the canonical form $T[(\mathcal{A}_n, c/\sqrt{n})_n]$ belongs to Class 1 if $\inf_n c_n > 0$ and to Class 2 if $\lim_n c_n = 0$.

In [23] a block sequence $(x_n)_{n \in \mathbb{N}}$ generating $\ell_p$-spreading model was constructed in Schlumprecht space $S$. This result combined with the result of I. Gasparis [19] led to the question if some biorthogonal sequence to $(x_n)_n$ generates a $c_0$-spreading model in $S^*$. We remark that this is not the case. In general, it is still unknown if any sequence in $S^*$ generates a $c_0$-spreading model. Finally we show that in (modified) mixed Tσiren spaces defined by $(\mathcal{S}_n)_n$ containing a block sequence generating $\ell_1^*$-spreading model there is a strictly singular non-compact operator on a subspace.
We describe now briefly the content of the paper. In the first section we recall the basic notions of the theory of mixed Tsirelson spaces and their modified versions, including the canonical representation of these spaces and the notion of a tree-analysis of a norming functional (Def. 1.8). The second section is devoted to the study of modified mixed Tsirelson spaces $T_M[\{S_n, \theta_n\}]$ satisfying the regularity condition. We extend the notion of an averaging tree (Def. 2.5) and present the notions of averages of different types, providing also upper (Lemma 2.10) and lower (Lemma 2.14) "Tsirelson-type" estimates. We conclude the section with the result on arbitrary distortion for spaces with $\theta_n/\theta^m \searrow 0$ (Theorem 2.19) and sequential minimality (Theorem 2.20). In the last section we study the existence of non-compact strictly singular operators in mixed Tsirelson spaces $T[\{A_n, \theta_n\}]$ (Theorem 3.4). We discuss the behaviour of a biorthogonal sequence to the sequence generating $\ell_1$-spreading model in Schauder space (Proposition 3.6) and the case of (modified) mixed Tsirelson spaces defined by families $\{S_n\}$ admitting $\ell_1^n$-spreading model (Theorem 3.8). We finish with the comments and questions concerning the Tsafiri space and richness of the set of subsymmetric sequences in a Banach space.

1. Preliminaries

We recall the basic definitions and standard notation.

By a tree we shall mean a non-empty partially ordered set $(T, \leq)$ for which the set $\{y \in T : y \leq x\}$ is linearly ordered and finite for each $x \in T$. If $T' \subseteq T$ then we say that $(T', \leq)$ is a subtree of $(T, \leq)$. The tree $T$ is called finite if the set $T$ is finite. The root is the smallest element of the tree (if it exists). The terminal nodes are the maximal elements. A branch in $T$ is a maximal linearly ordered set in $T$. The immediate successors of $x \in T$, denoted by $\text{succ}(x)$, are all the nodes $y \in T$ such that $x \leq y$ but there is no $z \in T$ with $x \leq z \leq y$. A node $x$ is a sibling of a node $y$, if $x, y \in \text{succ}(z)$ for some $z \in T$. If $x$ is a linear space, then a tree in $X$ is a tree whose nodes are vectors in $X$.

Let $X$ be a Banach space with a basis $(e_i)$. The support of a vector $x = \sum_i x_i e_i$ is the set $\text{supp} x = \{i \in \mathbb{N} : x_i \neq 0\}$, the range of $x$, denoted by $\text{range} x$, is the minimal interval containing $\text{supp} x$. Given any $x = \sum_i a_i e_i$ and finite $E \subseteq \mathbb{N}$, put $Ex = \sum_{i \in E} a_i e_i$. We write $x < y$ for vectors $x, y \in X$, if $\text{maxsupp} x < \text{minsupp} y$. A block sequence is any sequence $(x_i) \subseteq X$ satisfying $x_1 < x_2 < \ldots$. A closed subspace spanned by an infinite block sequence $(x_i)$ is called a block subspace and denoted by $[\{x_i\}]$.

**Notation 1.1.** Given any two vectors $x, y \in X$ we write $x \leq y$, if $\text{supp} x \subseteq \text{supp} y$, and we say that $x$ and $y$ are incomparable, if $\text{supp} x \cap \text{supp} y = \emptyset$.

Given a block sequence $(x_n) \subseteq X$ and a functional $f \in X^*$ we say that $f$ begins in $x_n$, if $\text{minsupp} f \in (\text{maxsupp} x_{n-1}, \text{maxsupp} x_n]$.

A basic sequence $(x_n)$ $C$-dominates a basic sequence $(y_n)$, $C \geq 1$, if for any scalars $(a_n)$ we have

$$\| \sum_n a_n y_n \| \leq C \| \sum_n a_n x_n \| .$$

Two basic sequences $(x_n)$ and $(y_n)$ are $C$-equivalent, $C \geq 1$, if $(x_n)$ $C$-dominates $(y_n)$ and $(y_n)$ $C$-dominates $(x_n)$.

We shall use the notions describing different ways of asymptotic representation of $\ell_p$, $1 \leq p < \infty$, and $c_0$ in a Banach space.

**Definition 1.2.** Let $E$ be a Banach space with a 1-subsymmetric basis $(u_n)$, i.e. 1-equivalent to any of its infinite subsequences. Let $(x_n)$ be a seminormalized basic sequence in a Banach space $X$. We say that $(x_n)$ generates $(u_n)$ as a spreading model, if for any $k \in \mathbb{N}$ and any $(a_i)_{i=1}^k \subseteq \mathbb{R}$ we have

$$\lim_{n_1 \to \infty} \lim_{n_2 \to \infty} \ldots \lim_{n_k \to \infty} \| \sum_{i=1}^k a_i x_{n_i} \|_X = \| \sum_{i=1}^k a_i u_i \|_E .$$
By [13] any seminormalized basic sequence admits a subsequence generating spreading model. We say that $(x_n)$ generates $\ell_p$- (resp. $c_0$-)spreading model, if $(u_n)$ is equivalent to the u.v.b. of $\ell_p$ (resp. $c_0$).

We say that a Banach space $X$ with a basis is $\ell_p$-asymptotic, $1 \leq p \leq \infty$, if any block sequence $n \leq x_1 < \cdots < x_n$ is $C$-equivalent to the u.v.b. of $\ell_p^n$, for any $n \in \mathbb{N}$ and some universal $C \geq 1$.

We work on two types of families of finite subsets of $\mathbb{N}$: $(A_n)_{n \in \mathbb{N}}$ and $(S_\alpha)_{\alpha < \omega_1}$. Let

$$ A_n = \{ F \subset \mathbb{N} : \# F \leq n \}, \quad n \in \mathbb{N}. $$

_Schreier families $(S_\alpha)_{\alpha < \omega_1}$, introduced in [1], are defined by induction:

$$ S_\alpha = \{ \{ k \} : k \in \mathbb{N} \} \cup \{ \emptyset \}, $$

$$ S_{\alpha + 1} = \{ F_1 \cup \cdots \cup F_k : k \leq F_1 < \cdots < F_k, f_1, \ldots, f_k \in S_\alpha \}, \quad \alpha < \omega_1. $$

If $\alpha$ is a limit ordinal, choose $\alpha_n \not\in \alpha$ and set

$$ S_\alpha = \{ F : F \in S_{\alpha_n} \text{ and } n \leq F \text{ for some } n \in \mathbb{N} \}. $$

Given a family $M = A_n$ or $S_\alpha$ we say that a sequence $E_1, \ldots, E_k$ of subsets of $\mathbb{N}$ is

1. $M$-admissible, if $E_1 < \cdots < E_k$ and $(\min E_i)_{i=1}^k \in M$,
2. $M$-allowable, if $(E_i)_{i=1}^k$ are pairwise disjoint and $(\min E_i)_{i=1}^k \in M$.

Let $X$ be a Banach space with a basis. We say that a sequence $x_1 < \cdots < x_n$ is $M$-admissible (resp. allowable), if $(\supp x_i)_{i=1}^n$ is $M$-admissible (resp. allowable).

**Definition 1.3 (Mixed and modified mixed Tsirelson space).** Fix a sequence of families $(M_n) = (A_{n_k})$ or $(S_{n_k})$ and sequence $C_n \subset (0, 1)$ with $\lim_{n \to \infty} c_n = 0$. Let $K \subset c_{00}$ be the smallest set satisfying the following:

1. $(\pm e_{n_k})_n \subset K$,
2. for any $f_1 < \cdots < f_k$ in $K$, if $(f_i)_{i=1}^k$ is $M_n$-admissible for some $n \in \mathbb{N}$, then $\theta_n(f_1 + \cdots + f_k) \in K$.

We define a norm on $c_{00}$ by $\| x \| = \sup \{ f(x) : f \in K \}, \quad x \in c_{00}.$ The mixed Tsirelson space $T(M_n, \theta_n)$ is the completion of $(c_{00}, \| \cdot \|)$.

The modified mixed Tsirelson space $T_M((M_n, \theta_n))$ is defined analogously, by replacing admissibility by allowability of the sequences.

It is standard to verify that the norm $\| \cdot \|$ defined above is the unique norm on $c_{00}$ satisfying the equation

$$ \| x \| = \max \left\{ \| x \|_\infty, \sup \left\{ \theta_n \sum_{i=1}^k \| E_i x \| : (E_i)_{i=1}^k \text{ is } M_n \text{-admissible (resp. allowable) } \right\}, \quad n \in \mathbb{N} \right\}. $$

It follows immediately that the u.v.b. $(e_n)$ is 1-unconditional in the space $T([M_n, \theta_n])$ and its modified version. It was proved in [7] that $T([S_{n_k}, \theta_n])$ is reflexive, also any $T([A_{n_k}, \theta_n])$ is reflexive, provided $\theta_n > 1/2$ for at least one $n \in \mathbb{N}$ [11].

Taking $M_n = M$ and $\theta_n = \theta$ for any $n$ we obtain the classical Tsirelson-type space $T[M, \theta]$. Recall that $T[A_n, \theta] = c_0$ if $\theta \leq 1/n$ and $T[A_n, \theta] = \ell_p$ if $\theta = 1/\sqrt{n}$ for $q$ satisfying $1/p + 1/q = 1$ [12, 11]. The space $T^*[1, 1/2]$ is the Tsirelson space.

Schlumprecht space $S$ is the space $T([A_n, \frac{1}{\log (n+1)}])$. Tsafri space is $T([A_n, \frac{1}{\sqrt{n}}])$ for $0 < c' < 1$. Modified Tsirelson-type spaces are isomorphic to their non-modified version [12, 15, 28], whereas the situation is quite different in mixed setting [9].

We present now the canonical form of a (modified) mixed Tsirelson space in both cases $M_n = A_{n_k}$ or $S_{n_k}, \quad n \in \mathbb{N}$.
Definition 1.4. [27] A mixed Tsirelson space $T(A_{n}, \theta_{n})_{n \in \mathbb{N}}$ is called a $p$-space, for $p \in [1, \infty)$, if there is a sequence $(p_{N})_{N} \subset (1, \infty)$ such that

1. $p_{N} \to p$ as $N \to \infty$, and $p_{N} \geq p_{N+1} > p$ for any $N \in \mathbb{N}$,
2. $T(A_{n}, \theta_{n})_{n=1}^{N}$ is isomorphic to $l_{p_{N}}$ for any $N \in \mathbb{N}$.

A $p$-space $T(A_{n}, \theta_{n})_{n \in \mathbb{N}}$ is called regular, if $\theta_{n} \downarrow 0$ and $\theta_{nm} \geq \theta_{n} \theta_{m}$ for any $n, m \in \mathbb{N}$. Recall that any $p$-space is isometric to a regular $p$-space [29].

Notation 1.5. Let $T(A_{n}, \theta_{n})_{n \in \mathbb{N}}$ be a regular $p$-space. If we set $\theta_{n} = 1/n^{1/n_{q}}$ with $q_{n} \in (1, \infty)$, $n \in \mathbb{N}$, then $q = \lim_{n} q_{n} = \sup_{n} q_{n} \in (0, \infty)$, where $1/p + 1/q = 1$, with usual convention $1/\infty = 0$.

In the situation as above let $c_{n} = \theta_{n}n^{1/q} \in (0, 1)$, $n \in \mathbb{N}$, if $p > 1$. To unify the notation put $c_{n} = \theta_{n}$, $n \in \mathbb{N}$, in case $p = 1$.

A space $T_{M}(S_{n}, \theta_{n})_{n \in \mathbb{N}}$ with $\theta_{n} \downarrow 0$ and $\theta_{nm} \geq \theta_{n} \theta_{m}$ is called a regular space. Notice that any modified mixed Tsirelson space $T_{M}(S_{k}, \theta_{n})_{n \in \mathbb{N}}$ is isometric to a regular modified mixed Tsirelson space (cf. [3]).

Notation 1.6. For a regular modified mixed Tsirelson space $T_{M}(S_{n}, \theta_{n})_{n}$ let $\theta = \lim_{n} \theta_{n}^{1/n} = \sup_{n} \theta_{n}^{1/n} \in (0, 1)$. We shall use also the following condition:

\[ \left\langle \alpha \right\rangle, \left( \theta_{n}/\theta^{n} \right)_{n} \downarrow \text{ i.e. } \theta_{n+m} \leq \theta_{n} \theta^{m} \text{ for any } n, m \in \mathbb{N}. \]

Given two families $\mathcal{M}, \mathcal{N}$ of finite subsets of $\mathbb{N}$ define

\[ \mathcal{M}[\mathcal{N}] = \{ F_{1} \cup \cdots \cup F_{k} : F_{1}, \ldots, F_{k} \in \mathcal{N}, (F_{1}, \ldots, F_{k}) \text{ \mathcal{M}-admissible, } k \in \mathbb{N} \}. \]

It follows straightforward that $S_{n}[S_{m}] = S_{n+m}$, for any $n, m \in \mathbb{N}$.

Lemma 1.7. The space $T_{M}(S_{n}[A_{2}], \theta_{n})_{n}$ is 3-isomorphic to $T_{M}(S_{n}, \theta_{n})_{n}$.

The proof of the above follows that of Lemma 4.5 [29] with "admissible" sequences replaced by "allowable" ones.

The following notion provides a useful tool for estimating norms in mixed Tsirelson spaces and their modified versions:

Definition 1.8. [The tree-analysis of a norming functional] Let $f \in K$, the norming set of $T(\mathcal{M}_{n}, \theta_{n})_{n}$ (resp. $T(\mathcal{M}_{n}, \theta_{n})_{n}$). By a tree-analysis of $f$ we mean a finite family $(f_{\alpha})_{\alpha \in \mathcal{T}}$ indexed by a tree $\mathcal{T}$ with a unique root $0 \in \mathcal{T}$ (the smallest element) such that the following hold

1. $f_{0} = f$ and $f_{\alpha} \in K$ for all $\alpha \in \mathcal{T}$,
2. $\alpha \in \mathcal{T}$ is maximal if and only if $f_{\alpha} = (\pm c_{n})$,
3. for every not maximal $\alpha \in \mathcal{T}$ there is some $n \in \mathbb{N}$ such that $(f_{\beta})_{\beta \in \text{succ}(\alpha)}$ is an $\mathcal{M}_{n}$-admissible (resp. allowable) sequence and $f_{\alpha} = \theta_{0}(\sum_{\beta \in \text{succ}(\alpha)} f_{\beta})$. We call $\theta_{0}$ the weight of $f_{\alpha}$.

For any $\alpha \in \mathcal{T}$, $\alpha > 0$, we define the tag $t(\alpha) = t(f_{\alpha})$ as $t(\alpha) = \prod_{\alpha > \beta \geq 0} \text{weight}(f_{\beta})$.

For any $\alpha \in \mathcal{T}$ we define also inductively the order of $\alpha$ as follows: $\text{ord}(0) = 0$ and for any $\beta \in \text{succ}(\alpha)$ we put $\text{ord}(\beta) = \text{ord}(\alpha) + n$, where $\text{weight}(f_{\alpha}) = \theta_{n}$.

Notice that every functional $f \in K$ admits a tree-analysis, not necessarily unique.

We shall use repeatedly the following

Remark 1.9. Let $X = T_{M}(S_{n}, \theta_{n})_{n}$ with $\left\langle \alpha \right\rangle$. Let $(f_{\alpha})_{\alpha \in \mathcal{T}}$ be a norming tree of a norming functional $f \in K$ and $\alpha \in \mathcal{T}$ not a terminal node. Let $f_{\alpha} = \theta_{0} \sum_{\beta \in \text{succ}(\alpha)} f_{\beta}$. Then by definition of Schreier families given any $k \in [\text{ord}(\alpha), \text{ord}(\alpha) + r_{n}]$ we can write $f_{\alpha}$ as follows

\[ f_{\alpha} = \theta_{r_{n}} \sum_{t \in A_{n}, s \in F_{t}} f_{s} \]
where \((f_\alpha)_{\alpha \in F}\) is \(S_{r_\alpha - (k-\text{ord}(\alpha))}\)-allowable, for any \(t \in A_\alpha\), and \((g_t)_{t \in A_\alpha}\) is \(S_{k-\text{ord}(\alpha)}\)-allowable, with \(g_t = \theta_{r_\alpha - (k-\text{ord}(\alpha))} \sum_{s \in F} f_t, t \in A_\alpha\). In particular for any \(f_\alpha\) and \(x \in X\) with non-negative coefficients we get by (\(\spadesuit\))

\[
    f_\alpha(x) = \theta_{r_\alpha} \sum_{t \in A_\alpha} g_t(x) \leq \theta_{k-\text{ord}(\alpha)} \sum_{t \in A_\alpha} g_t(x).
\]

As \(t(\alpha) \leq \text{ord}(\alpha) \leq \theta_{\text{ord}(\alpha)}\) it follows that \(t(\alpha) f_\alpha(x) \leq \theta_k \sum_{t \in A_\alpha} g_t(x)\).

2. Modified mixed Tsirelson spaces defined on Schreier families

In this section we present the main results on sequential minimality and arbitrary distortability of a regular modified mixed Tsirelson spaces \(T_M([S_\alpha, \theta_\alpha], n \in N]\) with (\(\spadesuit\)). In the first subsection we discuss the notion of an average, in the next two subsections we present "Tsirelson-type" estimations needed for the proof of main theorems in the last subsection. Since the u.v.b. in any (modified) mixed Tsirelson space and its dual is unconditional, we work in the sequel on functionals and vectors with non-negative coefficients. From now on we fix a regular modified mixed Tsirelson space \(X = T_M([S_\alpha, \theta_\alpha], n \in N]\).

2.1. Averages

In this part we recall the notion of an average [7] and present basic facts.

**Definition 2.1.** A vector \(x\) in a Banach space \(X\) with a basis is called an \((M, \varepsilon)\)-average of a block sequence \((x_i)_i < X, M \in N, \varepsilon > 0, if x = \sum_{i \in G} a_i x_i\) for some \(G \in S_M\) and \((a_i)_{i \in G} \subset (0, 1]\) with \(\sum_{i \in G} a_i = 1\) and for any \(F \in S_{M-1}\) we have \(\sum_{i \in F} a_i \leq \varepsilon\).

We shall use the following facts in the sequel.

**Fact 2.2.** (Lemma 4.9 [8]) Let \(x = \sum_{i \in F} a_i x_i\) be an \((M, \varepsilon)\)-average of normalized vectors \((x_i)_i \in F, M \in N, \varepsilon > 0\) and \(E\) an \(S_{M-1}\)-allowable family of sets. Let \(G = F \setminus K, where K = \{i \in F : \exists E \in E, E begins in x_i\}. Then for every \(i \in G\) the set \(\{Ex_i : E \in E, Ex_i \neq 0\}\) is \(S_1\)-allowable and

\[
    \sum_{E \in E} \|Ex_i\| \leq \sum_{E \in E} \|E(\sum_{i \in G} a_i x_i)\| + 2\varepsilon/\theta_M.
\]

**Fact 2.3.** Let \(x = \sum_{i \in F} a_i x_i\) be an \((M, \varepsilon)\)-average of normalized vectors \((x_i)_i \in F, M \in N, \varepsilon > 0\) and \(f\) a norming functional with a tree-analysis \((f_\alpha)_{\alpha \in T}\). Then there is subtree \(T'\) of \(T\) such that any terminal node of \(T'\) has order at least \(M\) and the functional \(f'\) defined by the tree-analysis \((f_\alpha)_{\alpha \in T'}\) satisfies \(f(x) \leq f'(x) + 2\varepsilon\).

**Proof.** Let \(E\) be the collection of all terminal nodes of \(T\) of order smaller than \(M\). Let \(G = \{i \in F : some f_\alpha begins in x_i, \alpha \in E\}. Since the set \((f_\alpha)_{\alpha \in E}\) is \(S_{M-1}\)-allowable, it follows \(G \setminus \{\min G\} \in S_{M-1}\) and \(f(\sum_{i \in G} a_i x_i) \leq a_{\min G} + \sum_{i \in C_\min G} a_i \leq 2\varepsilon\). We let \(T'\) be the tree \(T\) with removed nodes from the family \(E\). Then \(f(x) \leq f'(x) + f(\sum_{i \in G} a_i x_i) \leq f'(x) + 2\varepsilon\).

**Lemma 2.4.** Let \(X\) satisfy (\(\spadesuit\)). Let \(x = \sum_{i} a_i x_i\) be an \((M, \varepsilon)\)-average of a normalized block sequence \((x_i)_i \subset X, M \in N\). Then for any \(j \in N, j < M\) and \(S_j\)-allowable \((E_i)_i\) we have

\[
    \sum_{i} \|E_i x_i\| \leq \theta_1^{-1} \theta_{M-j-1} \sum_{i} a_i \|E_i x_i\| + 4\varepsilon/\theta_M.
\]

In particular \(\|x\| \leq \theta_1^{-1} \theta_{M-j-1} + 4\varepsilon/\theta_M\).

**Proof.** Take an \(S_j\)-allowable sequence \((E_i)_i\). For any \(l\) take a norming functional \(f_\alpha\) with \(\|E_l x\| = f_\alpha(x)\) and its tree-analysis \((f_\alpha)_\alpha \in T_l\). Let \(E\) be the collection of all terminal nodes \(\alpha \in T_l\) for all \(l\), such that \(\text{ord}_{T_l}(\alpha) \leq M - 1 - j\). Then the set \((f_\alpha)_\alpha \in E\) is \(S_{M-j}\)-allowable. By Fact 2.3 we can assume with error \(2\varepsilon\) that all terminal nodes of all \(T_j\) have order at least \(M - j\).
We will add in the tree-analysis \((f^t_{\alpha})_{\alpha \in T_l}\) additional nodes \((h_t)_l\) of order \(M - j - 1\), by grouping some of the nodes of \(T_l\), and by \(\bullet\) obtain the desired estimation.

For any \(l\) let \(E_l\) be collection of all \(\alpha \in T_l\) which are maximal with respect to the property \(\text{ord}_{T_l}(\alpha) \leq M - j - 1\). Fix \(\alpha \in E_l\). Then by the above reduction \(\alpha\) is not terminal, so \(f^t_{\alpha} = \theta x_i \sum_{\alpha \in \text{succ}(\alpha)} f^t_{\alpha}\) for some \(S_{\alpha}\)-allowable \((f^t_{\alpha})\). By Remark 19 for \(k = M - j - 1\) there are \(S_{M - j - 1 - \text{ord}(\alpha)}\) allowable functionals \((h_t)_{t \in A_{\alpha}}\) with

\[
 t(\alpha)f^t_{\alpha}(x) \leq \theta^{M-j-1} \sum_{t \in A_{\alpha}} h_t(x).
\]

It follows that \((h_t)_{t \in A_l}\) is \(S_{M-j-1}\)-allowable, where \(A_l = \cup_{\alpha \in E_l} A_{\alpha}\). Now we have

\[
\|E_l x\| = f_l(x) = \sum_{\alpha \in E_l} t(\alpha)f^t_{\alpha}(E_l x) \leq \sum_{\alpha \in E_l} \theta^{M-j-1} \sum_{t \in A_{\alpha}} h_t(E_l x) = \theta^{M-j-1} \sum_{t \in A_l} h_t(E_l x).
\]

Taking into account the error from erasing nodes with too small orders we obtain

\[
\sum_l \|E_l x\| \leq \theta^{M-j-1} \sum_l \sum_{t \in A_l} h_t(E_l x) + 2\varepsilon \leq \ldots
\]

Notice that \((h_t)_{t \in A}\) is \(S_{M-1}\)-allowable, where \(A = \cup_{l} A_l\). By Fact 2 with error \(2\varepsilon/\theta_M\) we assume that the family \((h_t(x_i))_{t, h_t(x_i) \neq 0}\) is \(S_{l}\)-allowable for each \(i\) and thus we have:

\[
\ldots \leq \theta^{M-j-1} \sum_l \sum_i a_i \sum_{t : h_t(x_i) \neq 0} h_t(E_l x_i) + 4\varepsilon/\theta_M
\]

\[
\leq \theta^{M-j-1}\theta_l^{-1} \sum_l \sum_i a_i \|E_l x_i\| + 4\varepsilon/\theta_M
\]

2.2. Averaging trees

In order to control the norm of splitting of a vector of special type into allowable, not only admissible parts, we compare it to the norm of splitting of a corresponding vector in the original Tsirelson space \(T[S_l, \theta]\). In this section we present the upper "Tsirelson-type" estimate for \((M, \varepsilon)\)-averages with more refined structure. We shall use the notion of an averaging admissible tree \([3]\) with additional features:

**Definition 2.5.** We call a tree \((x_i^{N_i})_{j=0}^{M} \subset X\) with weights \((N_i^{j})_{j=0}^{M} \subset \mathbb{N}\) and errors \((\varepsilon_i^{j})_{j=0}^{M} \subset (0, 1)\), an averaging tree, if

1. \((x_i^{j})_{i \in I_i}\) is a block sequence for any \(j, 1 = N^j M \leq \ldots \leq N^j 0\),
   Moreover for any \(j = 1, \ldots, M\) and \(i = 1, \ldots, N^j\) we have the following
2. there exists a nonempty interval \(I_i^{j} \subset \{1, \ldots, N^j - 1\}\) with \(#I_i^{j} = N_i^{j}\) such that \(\text{succ}(x_i^{j}) = (x_s^{j-1})_{s \in I_i^{j}}\),
3. \(x_i^{j} = 1/N_i^{j} \sum_{s \in I_i^{j}} x_s^{j-1}\),
4. \(2\varepsilon_i^{j} < N_i^{j} \leq \text{minsupp} x_i^{j}\),
5. \(\varepsilon_{i+1}^{j} < 1/(2^{j} \text{maxsupp} x_i^{j})\), \(\text{maxsupp} x_i^{j} < N_i^{j+1}\).

**Remark 2.6.** In the situation as above we define coefficients \((a_i^{j})_{j=0}^{M} \subset (0, 1]\), as satisfying \(x^M = \sum_{i=1}^{N^j} a_i^{j} x_i^{j}\). It follows straightforwardly that for any \(j = 0, \ldots, M\), \(i = 1, \ldots, N^j\) we have the following

6. \(\sum_{i=1}^{N^j} a_i^{j} = 1\),
7. \(a_i^{j} = \prod_{r=j+1}^{M} \frac{1}{N_i^{r}}\), where \(x_i^{r} \geq x_i^{j}\) for each \(M \geq r > j\),
8. \( a_i^0 = \sum_{m: x_m^0 \leq x_i^0} a_m^0. \)

Notice that any \( x_i^0 \) is a \((j, \varepsilon_j^0)\)-average of \((x_m^0)_{x_m^0 \leq x_i^0}.\)

To show the last statement notice that by (4) for any \( j, i \geq 1 \) the block sequence \( \text{succ}(x_j^0) \) is \( S_1 \)-admissible, thus any block sequence \((x_m^0)_{x_m^0 \leq x_i^0} \) is \( S_1 \)-admissible. To complete the proof notice that by the standard reasoning (cf. for example [30], last part of the proof of Proposition 3.6) we have the following fact:

**Fact.** Fix a block sequence \((x_m)_m \) and let \((x_m^0)_{m \in A_i} \) be a block sequence of \((M - 1, \varepsilon)\)-averages of \((x_m)_{m \in A_i} \) such that \( N > 2/\varepsilon \) and \( \varepsilon_{i+1} < 1/2^i \maxsupp x_i \). Then \( x = \frac{1}{N}(x_1 + \cdots + x_N) \) is a \((M, \varepsilon)\)-average of \((x_m)_{m \in A_i, i = 1, \ldots, N}.\)

The above Remark together with the construction of an averaging tree presented in [3] yields the standard

**Fact 2.7.** For any block sequence \((x_m)_m \) of \( X, \) any \( \varepsilon > 0 \) and any \( M \in \mathbb{N} \) there is an \((M, \varepsilon)\)-average \( x \) of \((x_m)\) with an averaging tree \((x_j^M)_{j=0, i=1}^{M,N} \) in \( X \) with suitable weights and errors \((\varepsilon_j^M)_{j=1, i=1}^{M,N},(\varepsilon_j^0)_{j=1, i=1}^{M,N} \) such that \( x_j^M = x, \varepsilon_1^0 = \varepsilon \) and \((x_m^0)_m \subset (x_m)_m \).

In order to deal with allowable splittings, we need the next result, stating - roughly speaking - that a restriction of an average \( x \) with an averaging tree high enough is still an average \( y \) with a strict control on the error on the new average \( y \) - depending on the error in the averaging tree of \( x \) corresponding to \( \minsupp y \).

**Lemma 2.8.** Let \((x_i^0), (N_i^0), (a_i^0), (\varepsilon_i^0)\) form an averaging tree for a \((M + \bar{M}, \varepsilon)\)-average \( x, \bar{M}, \varepsilon \in \mathbb{N}, \varepsilon > 0, \) of a normalized block sequence \((x_i^0), \) satisfying for any \( i, j \) the following

1. \( N_j^0 \geq 2^k_j \) for some \( k_j^0 \).
2. \( \varepsilon_i^0 \leq \theta_M \varepsilon /2, \varepsilon_i^0, i = 1, \ldots, N \) \( \maxsupp x_i \).

Then for any \( I \in \mathbb{N} \) with \( N_{\min I}^M \sum_{i \in I} a_i^0 x_i^M \) is a restriction of an \((M, \varepsilon_{\min I})\)-average of some block sequence \((y_i^0)_{i \in I} \) with \( \|y_i^0\| \leq 1 \) and such that the following property holds:

**(P)** For every \( k, i, l \) either \( x_i^0 \leq y_k^0 \) or \( x_i^0 \geq y_k^0 \) or \( x_i^0 \) and \( y_k^0 \) are incomparable, where \((y_i^0)_{i \in I}\) is the family of nodes of averaging tree of \( y \).

**Proof.** Let \( \varepsilon_i^0 = \varepsilon_{\min I}^M. \) We represent \( y = \sum_{i \in I} a_i^0 x_i^M \) as a restriction of an \((M, \varepsilon)\)-average. We construct inductively on \( l = M, M - 1, \ldots, 0 \) an averaging tree \((y_k^j)_{j=0, k=1}^{M,j} \) with weights \((W_k^j)\) and coefficients \((c_k^j)\), where \( y_k^j = 1/W_k^j \sum_{l \leq j} x_i^l \) and \( c_k^j = \prod_{k < k' \leq j} \frac{y_k^j}{y_k^{j+1}} \) such that \( y_k^j = y \) and the following is satisfied

\([P_0]\) \( c_k^j W_k^j = \sum_{m \in A_k^j} a_m^0 x_m^0 x_i^M \) for every \( k \) and \( l \leq M \).

\([P_1]\) For every \( k, i, l \) either \( x_i^l \leq y_k^j \) or \( x_i^l \) is incomparable with \( y_k^j \).

\([P_2]\) For every \( i, j, k, l \) either \( x_i^l \leq y_k^j \) or \( x_i^l \geq y_k^j \) or \( x_i^l \) and \( y_k^j \) are incomparable.

\([P_3]\) For every \( k, l \) we have \( W_k^l = \min_{l \leq k} W_k^l \), \( x_i^l \leq y_k^j \).

We allow one difference from the original definition: \# \( I^M = L = N_{\min I}^M \sum_{i \in I} a_i^0, \) not \( W_k^M \) otherwise \# \( J_k^M = W_k^M \) for any \( l < M \).

We let \( y_l^M = \sum_{i \in I^l} a_i^0 x_i^M = \sum_{m \in A_k^l} a_m^0 x_m^0 \) \( c_l^0 = 1, A_l^0 = A \) and \( W_l^0 = N_{\min I}^M \leq \minsupp y_l^M \). All properties \([P_0] - [P_3]\) are obviously satisfied.

Assume we have defined \((y_k^j)_{k,j} \) and \((x_i^l)_{i,l} \) for some \( M \geq l > 2 \) satisfying the above. Fix \( k \) and consider \( A_k^l \). Pick any \( m \in A_k^l \). By \( [P_1] \) in inductive assumption we have \( x_i^l \leq y_k^j \) for any \( l \leq r \leq M, i, r, k \) with \( x_m^0 \leq x_i^l \) and \( x_m^0 \leq y_k^j \). Thus \( N_{\min}^L \geq W_k^l \) for any \( l \leq r \leq M, i, r, k \) as above. By Remark 2.6 and \([P_3]\) we have

\[ a_i^0 = \prod_{r=1}^M \frac{1}{W_r^l} \leq \frac{1}{N_r^l} \leq \prod_{r=1}^M \frac{1}{W_r^k} = \frac{c_k^j}{W_k^j}. \]
Recall that all coefficients \( a_m^0, c^l_k W_k^l \) are some powers of \( 1/2 \) and \((a_m^0)_m\) is non-increasing. Moreover for \( l < M \) we have \( \sum_{m \in A_k} a_m^0 = c^l_k \), hence we can split \( A_k^l \) into \( W_k^l \)-many successive sets \((A_{s-1}^l W_k^l)_{s=1}^L \) such that for each \( s \) we have
\[
\sum_{m \in A_{s-1}^l} a_m^0 = \frac{c^l_k}{W_k^l}.
\]
In case \( l = M \) we have \( \sum_{m \in A_k^M} a_m^0 = L/W_k^M \), hence we can split \( A_k^M \) into \( L \)-many sets \((A_{s-1}^M W_k^M)_{s=1}^L \) such that for each \( s \) we have
\[
\sum_{m \in A_{s-1}^M} a_m^0 = \frac{c^M_k}{W_k^M} = \frac{1}{W_k^M}.
\]
We define then \((y_{s-1}^l)_s\) and \((c^{-1}_s)_s\) by
\[
\frac{c^l_k}{W_k^l} y_{s-1}^l = \sum_{m \in A_{s-1}^l} a_m^0 x_m^l, \quad c^{-1}_s = \frac{c^l_k}{W_k^l}.
\]
Hence obviously \( y_{s-1}^l = 1/W_k^l \sum_{m \in A_{s-1}^l} a_m^0 \). We let also \( W_{s-1}^l = \min\{A_{s-1}^l : y_{s-1}^l \leq y_{l-1}^l \} \) and thus we finish construction of vectors on level \( l - 1 \) satisfying (P0) and (P3).
Now we verify property (P1). Notice that by property (P1) on level \( l \) for each \( k \) we have \( \supp y_k^l = \cup\{\supp x_k^l : x_k^l \leq y_k^l \} = \cup\{\supp x_{s-1}^l : x_{s-1}^l \leq y_{k-1}^l \} \). In case \( l < M \) by Remark 2.6 and (P0) for \( l \) we have
\[
\sum_{r : y_{r-1}^l \leq y_k^l} c^{-1}_r W_k^l = c^{-1}_k = \sum_{m \in A_k^l} a_m^0 = \sum_{s : x_{s-1}^l \leq y_{k-1}^l} a_s^{-1},
\]
and as the construction each \( a_{s-1}^l \leq c_k^l / W_k^l = c^{-1}_r \). In case of \( l = M \) we have
\[
\sum_{r : y_{r-1}^M \leq y_k^M} c^{-1}_r W_k^M = L = \sum_{m \in A_k^M} a_m^0 = \sum_{s : x_{s-1}^M \leq y_{k-1}^M} a_s^{-1},
\]
and each \( a_{s-1}^M \leq 1/W_k^M = c^{-1}_k W_k^l \). Since all coefficients are powers of \( 1/2 \) and the sequence \((a_{s-1}^l)_s\) is non-increasing we can partition the set \( \{ s : x_{s-1}^l \leq y_k^l \} \) into \( \cup\{B_r : y_{r-1}^l \leq y_k^l \} \) such that for any \( r \) we have \( c^{-1}_r = \sum_{s \in B_r} a_s^{-1} \). Consequently for any \( y_{r-1}^l \leq y_k^l \) and \( x_{s-1}^l \leq y_{k-1}^l \) we have either \( y_{r-1}^l \leq x_{s-1}^l \) or \( y_{r-1}^l \) and \( x_{s-1}^l \) are incomparable.

The property (P2) is verified analogously by induction. If for some \( l, j, k \) we have \( \supp y_k^l = \cup\{\supp x_j^l : x_j^l \leq y_k^l \} \), then we show that for any \( y_{j-1}^l \leq y_k^l \) and \( x_{s-1}^l \leq y_{k-1}^l \) we have either \( y_{j-1}^l \leq x_{s-1}^l \) or \( y_{j-1}^l \) and \( x_{s-1}^l \) are incomparable. The same argument works if \( \supp x_j^l = \cup\{\supp y_k^l : x_j^l \leq y_k^l \} \) for some \( i, j, l \).

Define the error \( \delta_k^l \) for each \( l = M, \ldots, 1 \) and \( k = 1, \ldots, K_l \). For \( k = 1 \) and any \( l = M, \ldots, 1 \) let \( \delta_1^l = \varepsilon_1 \).

By property (P1) for any \( l, k \) there is some \( i_k \geq k \) with
\[
\maxsupp y_k^l \leq \maxsupp x_{i_k}^l < \minsupp x_{i_{k+1}}^l \leq \minsupp y_{k+1}^l.
\]
Let \( \delta_{k+1}^l = \varepsilon_{i_{k+1}}^l \) for any \( k \geq 1 \). We verify condition (5) of Definition 2.5. For \( k = 1 \) and \( l = M, \ldots, 1 \) we have
\[
W_k^l \geq N_{\min l}^M \geq 2/e_{\min l}^M = 2/\delta_1^l.
\]
On the other hand we have for any \( l = M - 1, \ldots, 1 \) and \( k = 1, \ldots, K_l - 1 \)
\[
\delta_{k+1}^l = \varepsilon_{i_{k+1}}^l < 1/2^{i_{k+1}} \maxsupp x_{i_k}^l \leq 1/2^{k} \maxsupp y_k^l,
\]
and
\[
W_{k+1}^l \geq N_{i_{k+1}}^l > 2/e_{i_{k+1}}^l = 2/\delta_{k+1}^l.
\]
Hence \( (y_k^l)_{k,l}, (W_k^l)_{k,l}, (c_k^l)_{k,l}, (\delta_k^l)_{k,l} \) form an averaging tree and thus \( y \) is \((M, \varepsilon_1)\)-average of \((y_k^l)_{k} \).
Notice that
\[
\| c_k^l y_k^l \| = \| \sum_{m \in A_k^l} a_m^0 x_m^l \| \leq \sum_{m \in A_k^l} a_m^0 = c_k^l,
\]
therefore \( \| y_k^l \| \leq 1 \). Moreover property (P2) includes property (P).
Remark 2.9. Note that by the construction each sequence \((y^k_i)_{x \in E}\) is \(S_1\)-admissible for any \(k, l\). Hence it readily follows that for every set \(F\) of incomparable nodes \((y^l_k)\) the functional \(\sum_{y^l_k \in F} \theta^{M-1} x^M_{\minsupp y^l_k}\) is a norming functional on the space \(T[S_1, \theta]\).

Note also that for any \(k_0, l_0\) the family \(\{x^M : x^M_i \leq y^l_{k_0}, l_0 = \min\{m \geq l \geq 0 : x^M_i < y^l_k\text{ for some }k\}\}\) is \(S_1\)-admissible.

The next Lemma provides a "Tsirelson-type" upper estimate for the norms of averages.

**Lemma 2.10.** Let \((x^M_i), (N^l_j), (a^i_k), (\epsilon^i_k)\) form an averaging tree for a \((2M - 3, \epsilon)\)-average \(x, M > 1, \epsilon > 0\), of a normalized block sequence \((x^0_i)_i\), satisfying for any \(i, j\) the following:

1. \(N^l_j = 2^k_l\) for some \(k^l_j\),
2. \(\epsilon^i_k \leq \theta M \epsilon/2, \epsilon^{i+1}_k \leq \theta M \epsilon/2^M\) maxsupp \(x^i_k\).

Fix an \(S_{M-4}\)-allowable family \(E\) of subsets of \(\mathbb{N}\), such that the family \(\{E \in E : E_{x^M} \neq 0\}\) is \(S_1\)-allowable for any \(i\), and coefficients \((t_E)_{E \in E} \subset [0, 1]\).

Then there is a partition \((V_E)_{E \in E}\) of nodes \((x^0_i)_i\), with minsupp \(x^0_{\min V_k} \geq M\) \(E\), such that

\[
\sum_{E \in E} t_E \|E x\| \leq C \sum_{E \in E} t_E \sum_{i \in V_k} a^0_i \|x^0_i\|_{T[S_1, \theta]} + C \epsilon
\]

for some universal constant \(C\) depending only on \(\theta_1\) and \(\theta_1\).

**Proof.** **STEP 1.** Let us recall that \(x\) is an \((M - 3, \epsilon)\)-average of \((x^M_i)_i\). First let \(E_i = \{E \in E : E \text{ begins at } x^M_i\}\) and \(J = \{i : E_i \neq 0\}\). As \((x^M_i)_{i \in J, \min J}\) is \(S_{M-4}\)-admissible, we have

\[
\sum_{E \in E} t_E \|E x\| \sum_{i \in J} a^M_i \|x^M_i\| \leq \sum_{i \in J} a^M_i \sum_{E \in E} \|E x^M_i\| \leq \theta_1^{-1} \sum_{i \in J} a^M_i \leq \theta^{-1} \epsilon.
\]

For any \(E \in E\) let \(I_E = \{i : E \neq 0\}, i_E = \min I_E\) and \(\epsilon_E = \epsilon^M_{i_E}\). Compute

\[
\sum_{E \in E} \epsilon_E \leq \epsilon \theta_M \sum_{i \in J} \maxsupp x^M_{i-1} \leq \epsilon \theta_M \sum_{i \in J} \maxsupp x^M_i \leq \epsilon \theta_M.
\]

**STEP 2.** Fix \(E \in E\). Let \(\sum_{i \in I_E} a^M_i x^M_i = \sum_{m \in K} a^0_m x^0_m\). Notice that each \(a^0_m \leq 1/N^M_{i_E}\) and \((a^0_m)_{m}\) is non-increasing, therefore we can partition \(K\) into intervals \(A < B\) with \(\sum_{m \in A} a^0_m = L/N^M_{i_E}\) and \(\sum_{m \in B} a^0_m = \delta/N^M_{i_E}\) for some \(L \in N\) and \(0 < \delta < 1\). Hence we can erase \(\sum_{m \in E} a^0_m x^0_m\) with error \(\delta/N^M_{i_E} \leq 1/N^M_{i_E} \leq \epsilon\).

After this reduction by Lemma 2.8 the vector \(y = \sum_{i \in I_E} a^M_i x^M_i\) is a restriction of an \((M - 2, \epsilon)\)-average \(\sum_k c^2_k y^2_k\) with \(\|y^2\| \leq 1\) and property (P) given by a suitable averaging tree \((y^k_k)_{k, l}\) with proper weights, coefficients and errors.

We take the family \(K = \{k : \minsupp x^M_i \in y^k_k\}\). Since \((x^M_i)\) is an \(S_{M-3}\)-admissible family and \(y\) is an \((M - 2, \epsilon)\)-average of \((y^k_k)\), we can erase \(\sum_{k \in K} c^2_k y^2_k\) with error \(2 \epsilon\). For any \(i\) let

\[
l_{E, i} = \min\{M \geq l \geq 0 : y^l_k \geq x^M_i\} \text{ for some } k.
\]

By the above reduction and (P) we can assume that \(l_{E, i} \geq 2\) for all \(i \in I_E\). Let

\[
K_{E, i} = \{k : y^k_k \geq x^M_i\} \text{ for any } i \in I_E.
\]
Compute by Lemma 2.4 for the \((M - 2, \varepsilon_E)\)-average \(\sum_k c_k^2 y_k^2\) and \(j = 0\)
\[
\|Ex\| = \|E\sum_k c_k^2 y_k^2\| \leq \\|E\sum_k c_k^2 E y_k^2\| + 2\varepsilon_E \\
\leq \theta_1^{-1}\theta^{M-3} \sum_{i \in I_E} \sum_{k \in K_E,i} c_k^2 \|E y_k^2\| + 6\varepsilon_E/\theta_M \\
= \theta_1^{-1}\theta^{M-3} \sum_{i \in I_E} a_i^M \sum_{k \in K_E,i} \frac{c_k^2}{a_i^M} \|E y_k^2\| + 6\varepsilon_E/\theta_M.
\]

**STEP 3.** Fix \(i \notin J\). Put \(F_i = \{E \in \mathcal{E} : i \in I_E\} = \{E \in \mathcal{E} : Ex_i^M \neq 0\}\). For any \(E \in F_i\) and \(k \in K_{E,i}\), let \(w_k = \frac{c_k^2}{a_i^M} E y_k^2\). Notice that \((w_k)_k\) is a partition of \(x_i^M\). For each \(k \in K_{E,i}\) take the norming functional \(f_k\) with \(f_k(w_k) = \|w_k\|\) and \(\text{supp}\ f_k \subset \text{supp}\ w_k\).

We gather all the terminal nodes in the tree-analysis of \(f_k\) for all \(k \in K_{E,i}\), \(E \in F_i\), of order smaller than \(M - l_{E,i}\). By the assumption on \(\mathcal{E}\) and the fact that \(l_{E,i} \geq 2\) they form an \(\mathcal{S}_{M-1}\)-allowable family, hence as \(x_i^M\) is an \((M, \varepsilon_i^M)\)-average, we can erase these nodes with total error \(2\varepsilon_i^M\).

By Remark 1.9, adding nodes in the tree-analysis of each \(f_k, k \in K_{E,i}\), on the level \(M - l_{E,i}\), we get \(\|w_k\| \leq \theta^{M-l_{E,i}} \sum_l f_l^1(x_l^M)\) for some \(\mathcal{S}_{M-1}\)-allowable functionals \((f_l^1)_l\). Pick \(E_i\) with \(t_{E_i, \theta^{-1}\varepsilon_i} = \max\{t_{E, \theta^{-1}\varepsilon_i} : E \in F_i\}\). Let \(l_i = l_{E_i,i}\) and compute
\[
\sum_{E \in F_i} t_{E_i} \sum_{k \in K_{E,i}} \|\frac{c_k^2}{a_i^M} E y_k^2\| \leq \sum_{E \in F_i} t_{E_i} \theta^{M-l_{E,i}} \sum_{k \in K_{E,i}} \sum_l f_l^1(x_l^M) + 2\varepsilon_i^M \leq \ldots
\]

Notice again that \((f_l^1)_{l,k \in K_{E,i}, E \in F_i}\) is an \(\mathcal{S}_{M-1}\)-allowable family (as before by \(l_{E,i} \geq 2\) and assumption on \(\mathcal{E}\)).

As \(x_i^M\) is an \((M, \varepsilon_i^M)\)-average of suitable \((x'_0)_m\), by Fact 2.2 with error \(2\varepsilon_i^M/\theta_M\), we may assume that for any \(m\) the family \((\text{supp\ } f_k^1 \cap \text{supp\ } x'_0_{m,k}, k \in K_{E,i}, E \in F_i\) is \(\mathcal{S}_1\)-allowable. Therefore we continue the estimation
\[
\ldots \leq t_{E_i} \theta^{M-l_i} \sum_{E \in F_i} \sum_{k \in K_{E,i}} \sum_l f_l^1(x_l^M) + 4\varepsilon_i^M/\theta_M \leq \theta_1^{-1} t_{E_i} \theta^{M-l_i} + 4\varepsilon_i^M/\theta_M.
\]

**STEP 4.** We define \(J_E = \{i : E = E_i\} \subset I_E\) for any \(E \in \mathcal{E}\). Notice that \((J_E)_{E \in \mathcal{E}}\) are pairwise disjoint and compute, using the previous steps
\[
\sum_{E \in J_E} t_{E} \|Ex\| \leq \sum_{E \in J_E} t_{E} \|E \sum_{i \in J} a_i^M x_i^M\| + \sum_{E \in J_E} t_{E} \|E \sum_{i \in J_E} a_i^M x_i^M\| \quad \text{(STEP 1)} \\
\leq 2\theta_1^{-1} \varepsilon + \theta_1^{-1} \theta^{M-3} \sum_{i \in J} \sum_{E \in J_E} a_i^M \sum_{k \in K_{E,i}} t_{E} \|\frac{c_k^2}{a_i^M} E y_k^2\| + 6 \sum_{E \in J_E} \varepsilon/\theta_M \quad \text{(STEP 2)} \\
\leq \theta_1^{-2} \theta^{M-3} \sum_{i \in J} \sum_{E \in J_E} a_i^M \sum_{k \in K_{E,i}} t_{E} \|\frac{c_k^2}{a_i^M} E y_k^2\| + (6 + 2\theta_1^{-1})\varepsilon \\
\leq \theta_1^{-2} \theta^{M-3} \sum_{i \in J} \sum_{E \in J_E} a_i^M \theta^{M-l_i} + 4 \sum_{i \in J} \varepsilon_i^M/\theta_M + (6 + 2\theta_1^{-1})\varepsilon \quad \text{(STEP 3)} \\
\leq \theta_1^{-2} \theta^{M-3} \sum_{i \in J_E} t_{E} \sum_{i \in J_E} a_i^M \theta^{M-l_i} + (10 + 2\theta_1^{-1})\varepsilon \leq \ldots
\]

By Remark 2.9 for any \(E \in \mathcal{E}\) the formula \(\sum_{i \in J_E} \theta^{M-l_i+1} e_{\min\text{supp}\ x_i^M}\) defines a norming functional in \(T[S_1, \theta]\). Therefore for any \(E \in \mathcal{E}\) we have
\[
\sum_{i \in J_E} a_i^M \theta^{M-l_i} \leq \theta^{-1} \|\sum_{i \in J_E} a_i^M e_{\min\text{supp}\ x_i^M}\| T[S_1, \theta],
\]
and we continue the above estimation

\[ \ldots \leq \theta_1^{-2} \theta^{-4} \sum_{E \in \mathcal{E}} t_E \left\| \sum_{i \in J_E} a_i^M e_{\minsupp x_i} \right\|_{T[S_i, \theta]} + (10 + 2 \theta_1^{-1}) \varepsilon \leq \ldots \]

Consider \( z_i^M = 1/a_i^M \sum_{x_i^0 \leq x_i^M} a_i^0 e_{\minsupp x_i} \), for \( i = 1, \ldots, N^M \), which are \((M, \varepsilon_i^M)\)-averages in \( T[S_i, \theta] \) by Remark 2.6. As \( \|z_i^M\|_{T[S_i, \theta]} \geq \theta^M \) for each \( i \), we continue

\[ \ldots \leq \theta_1^{-2} \theta^{-4} \sum_{E \in \mathcal{E}} t_E \left\| \sum_{i \in J_E} a_i^M z_i^M \right\|_{T[S_i, \theta]} + (10 + 2 \theta_1^{-1}) \varepsilon \]

\[ \leq \theta_1^{-2} \theta^{-4} \sum_{E \in \mathcal{E}} t_E \left( \sum_{i \in J_E} a_i^0 e_{\minsupp x_i^0} \right) \|T[S_i, \theta]\| + C \varepsilon, \]

which ends the proof with \( C = 10 + 2 \theta_1^{-2} \theta^{-4} \) and \( V_E = \{ m : x_i^0 \leq x_i^M, i \in J_E \} \) for each \( E \in \mathcal{E} \).

2.3. Special types of averages

For the rest of this subsection we assume that the considered regular modified Tsirelson space \( X = T_M ([S_n, \theta_n]) \) satisfies (\( \mathcal{A} \)). In this setting we present the lower "Tsirelson-type" estimate, using special types of averages. We start with Corollary 4.10 [29] recalled below.

**Proposition 2.11.** For any block subspace \( Y \) of \( X \), any \( M \in \mathbb{N} \) and \( 0 \leq \varepsilon < \theta M \theta^M / 4 \), there is an \((M, \varepsilon)\)-average \( x \in Y \) of some normalized block sequence in \( Y \) such that

\[ \theta^M D \geq \sup \left\{ \left\| E_i x \right\| : \right. \right. \left. S_i \text{-allowable } (E_i) \right\} \geq \theta^{M-j}/D \]

for any \( 0 \leq j \leq M \) and some universal constant \( D \) depending only on \( \theta_1 \) and \( \theta \).

**Proof.** We recall Lemma 4.9 [29], whose proof is valid, line by line, also in the modified case. Lemma 4.9 [29] and Lemma 2.4 yield the Proposition.

**Definition 2.12.** A special \((M, \varepsilon)\)-average \( x \), \( M \in \mathbb{N}, \varepsilon > 0 \), is any \((M, \varepsilon)\)-average satisfying assertion of Proposition 2.11.

For the next lemma we shall need the following observation.

**Fact 2.13.** Fix \( M \in \mathbb{N} \). Then for any \( G \in S_M \) and any \( z = \sum_{i \in G} a_i e_i \in T[S_1, \theta], (a_i)_{i \in G} \subset [0, 1] \), there is a norming functional \( f \) with a tree-analysis with height at most \( M \), such that \( \|z\|_{T[S_1, \theta]} \leq 2f(z) \).

**Proof.** Take a norming functional \( g \) with a tree-analysis \((g_i)_{i \in T}\) satisfying \( g(z) = \|z\|_{T[S_1, \theta]} \). Let \( I \) be the set of all terminal nodes of \( T \) with order at most \( M \) and let \( g_1 \) be the restriction of \( g \) to \( I \) and let \( g_2 = g - g_1 \). If \( g_1(z) \geq g_2(z) \) then we let \( f = g_1 \). Assume that \( g_1(z) \leq g_2(z) \) and compute

\[ g(z) \leq 2g_2(z) \leq 2\theta^{M+1} \sum_{i \in G \setminus I} a_i \leq 2\theta^M \sum_{i \in G} a_i = 2f(z), \]

where \( f = \theta^M \sum_{i \in G} e_i \), which ends the proof.

The major obstacle in obtaining the lower "Tsirelson-type" estimate for norm is the fact that given an \((M, \varepsilon)\)-average \( x = \sum_{i \in F} a_i x_i \), we do not control the norm of \( \sum_{i \in G} a_i x_i \) \( \subset F \), in general case. The next result provides a block sequence \((x_i)\) for which \( S_M \)-admissible subsequence dominates suitable subsequence of the basis in the original Tsirelson space. This result is a generalization in the setting of mixed Tsirelson spaces of Prop. 3.3 [8].
Lemma 2.14. For every block subspace $Y$ of $X$ and every $M \in \mathbb{N}$, $\delta > 0$, there exists a block sequence $(x_i)$ of $Y$ satisfying for any $G \in S_M$ and scalars $(a_i)_{i \in G}$

$$
\| \sum_{i \in G} a_i x_i \| \geq \frac{1}{2} (1 - \delta) \| \sum_{i \in G} a_i \|_{\text{minsupp}(x_i, \|T[S_i, \theta]\|)}.
$$

(2.1)

Proof. Assume the contrary. Notice first that for any $M \in \mathbb{N}$ we have

$$
(\sqrt[2]{\theta_m})^M \leq \sqrt[2]{\theta_{m+1}} \leq \sqrt[2]{\theta_m^M},
$$

thus $\lim_{m \to \infty} \sqrt[2]{\theta_{m+1}} = \theta^M$. Pick $m \in \mathbb{N}$ such that $\sqrt[2]{\theta_{m+1}} > \sqrt[2]{D^2(1 - \delta)\theta^M}$ with $D$ as in Prop. 2.11. Take a block sequence $(x^0_i)$ of special $(M, \delta)$-averages, for some $\varepsilon > 0$.

Since (2.1) fails there is an infinite sequence $G_k$ of successive elements of $S_M$ and coefficients $(a^1_i)_{i \in G_k}$ such that

$$
\| \sum_{i \in G_k} a^1_i x^0_i \| < \frac{1}{2} (1 - \delta) \| \sum_{i \in G_k} a^1_i \|_{\text{minsupp}(x^0_i, \|T[S_i, \theta]\|)},
$$

where $m^0 = \text{minsupp}(x^0_i)$ for each $i$. Set $x^1_i = \sum_{i \in G_k} a^1_i x^0_i$, $k \in \mathbb{N}$, and by Fact 2.13 take norming functionals $f^1_k$ of the space $T[S_1, \theta]$ with a tree-analysis of height at most $M$ with

$$
\| \sum_{i \in G_k} a^1_i \|_{\text{minsupp}(x^0_i, \|T[S_i, \theta]\|)} \leq 2 f_k \left( \sum_{i \in G_k} a^1_i \|x^0_i\|_{\text{minsupp}(x^0_i, \|T[S_i, \theta]\|)} \right).
$$

Assume that we have defined $(x^{j-1}_k)_k$ and $(f^{j-1}_k)_k$ for some $j < m$. Then the failure of (2.1) implies the existence of a sequence $(G^j_k)_k$ of successive elements of $S_M$ and a sequence $(a^j_i)_{i \in G^j_k}$ such that

$$
\| \sum_{i \in G^j_k} a^j_i x^{j-1}_i \| < \frac{1}{2} (1 - \delta) \| \sum_{i \in G^j_k} a^j_i \|_{\text{minsupp}(x^{j-1}_i, \|T[S_i, \theta]\|)},
$$

where $m^{j-1} = \text{minsupp}(x^{j-1}_i)$. Set $x^{j}_i = \sum_{i \in G^j_k} a^j_i x^{j-1}_i$, for $k \in \mathbb{N}$, and take norming trees $f^j_k$ of the space $T[S_1, \theta]$ with a tree-analysis of height at most $M$ such that

$$
\| \sum_{i \in G^j_k} a^j_i \|_{\text{minsupp}(x^{j-1}_i, \|T[S_i, \theta]\|)} \leq 2 f_k \left( \sum_{i \in G^j_k} a^j_i \|x^{j-1}_i\|_{\text{minsupp}(x^{j-1}_i, \|T[S_i, \theta]\|)} \right).
$$

The inductive construction ends once we get the vector $x^m_i$ and the functional $f^m_i$.

Each functional $f^m_i$ is of the form $\sum_{i \in G^m_k} \theta^i e^*_i$, by construction satisfying

$$
\|x^m_i\| < (1 - \delta) \sum_{i \in G^m_k} \theta^i \|x^{j-1}_i\|.
$$

(2.2)

Inductively, beginning from $f^m_i$, we produce a tree-analysis of some norming functional $f$ on $T[S_1, \theta]$ by substituting each terminal node $e^*_i$, $j = 1, \ldots, m$, by the tree-analysis of the functional $f^j_k$.

Put $G = \bigcup_{k \in \mathbb{N}} G^k$, $j = \sum_{i \in G^k} \theta^i e^*_i$. Notice that $l_i \leq mM$ for any $i \in G$, as the height of each $f^j_k$ does not exceed $M$. We compute the norm of $x^m_i$, which is of the form

$$
x^m_i = \sum_{k \in \mathbb{N}} a^m_{ik} \cdots \sum_{j \in G^j_k} a^m_{jk} \cdots \sum_{i \in G^0_{k_1}} a^m_{i_1} x^0_i = \sum_{i \in G} b_i x^0_i.
$$
Since each \( x_i^0 \) is a special \((m, \varepsilon, \theta)\)-average, for some \( S_{m, \varepsilon} \)-allowable sequence \((E_l)_{l \in L_i}\), we have \( \|x_i^0\| \leq D^2 \varepsilon M - \varepsilon \|E_l x_i^0\| \).

We have on one hand by repeated use of (2.2)

\[
\|x_i^n\| \leq (1 - \delta)^n \sum_{i \in G} \theta^l b_i \|x_i^0\|
\]

\[
\leq (1 - \delta)^n D^2 \sum_{i \in G} \theta^l b_i \varepsilon M - \varepsilon \|E_l x_i^0\|
\]

\[
= (1 - \delta)^n D^2 \varepsilon M \sum_{i \in G} b_i \|E_l x_i^0\|.
\]

On the other hand notice that \((E_l)_{l \in G}\) is \( S_{m, \varepsilon} \)-allowable by the definition of \( f \) and \((l_i)_{i \in G}\); thus

\[
\|x_i^n\| \geq \theta_m M \sum_{i \in G} b_i \|E_l x_i^0\|,
\]

which brings \( \theta_m M \leq (1 - \delta)^n D^2 \varepsilon M \), a contradiction with the choice of \( m \).

**Definition 2.15.** A Tsirelson \((M, \varepsilon)\)-average \( x \in X \), \( M \in \mathbb{N} \), \( \varepsilon > 0 \), is an \((M, \varepsilon)\)-average \( x = \sum_{i \in F} a_i x_i \) of a normalized block sequence \((x_i)\) satisfying the assertion of the Lemma 2.14 with \( \delta = 1/2 \).

Notice that by Lemma 2.4 every Tsirelson average is also a special average (with a possibly different constant).

**Definition 2.16.** A RIS of (special, Tsirelson) averages is any block sequence of (special, Tsirelson) \((n_k, \varepsilon/2^k)\)-averages \((x_k) \subset X\) for \( \varepsilon > 0 \) and \((n_k)_k \subset \mathbb{N}\) satisfying

\[
\theta_{k+1} \|x_k\| \leq \frac{\varepsilon}{2^{k+1}}, \quad k \in \mathbb{N},
\]

where \( l_k = \max \{ l \in \mathbb{N} : 4l \leq n_k \} \), \( k \in \mathbb{N} \).

We need the following technical lemma, mostly reformulating Lemma 7 [22]:

**Fact 2.17.** Take RIS of normalized averages \((x_k)\), for some \((n_k) \subset \mathbb{N} \) and \( \varepsilon > 0 \), and some \( x = \sum_k b_k x_k \) with \((b_k) \subset [0, 1]\). Then for any norming functional \( f \) with a tree-analysis \((f_\alpha)_{\alpha \in T}\) there is a subtree \( T' \) such that the corresponding functional \( f' \) defined by the tree-analysis \((f_\alpha)_{\alpha \in T}\) satisfies \( f(x) \leq f'(x) + 3\varepsilon \) and the following holds for any \( k \)

(a) any node \( \alpha \) of \( T' \) with \( f_\alpha(x_k) \neq 0 \) satisfies \( \text{ord}(\alpha) < n_k + 1/4 \),

(b) any terminal node \( \alpha \) of \( T' \) with \( f_\alpha(x_k) \neq 0 \) satisfies \( \text{ord}(\alpha) \geq n_k \).

**Proof.** In order to prove (a) we repeat the reasoning from the proof of Lemma 7 [22]. For any \( k \) let \( F_k \) be the collection of all nodes in \( T \) which are minimal with respect to the property \( \text{ord}(\alpha) \geq n_{k+1}/4 \) and \( f_\alpha(x_k) \neq 0 \). Then

\[
\sum_{\alpha \in F_k} t(\alpha) f_\alpha(x_k) \leq \theta_{k+1} \|x_k\| l_i \leq \frac{\varepsilon}{2^{k+1}}.
\]

Thus we can erase all nodes from \( F_k \) restricted to supports of \( x_k \), for all \( k \), with error \( \sum_k b_k \frac{\varepsilon}{2^{k+1}} \leq \varepsilon \).

For (b) we use Fact 2.3 for erasing all terminal nodes \( \alpha \) of \( T \) with \( f_\alpha(x_k) \neq 0 \) with error \( 2\varepsilon_k \), for any \( k \).

**Lemma 2.18.** Let \( x = \sum_k a_k x_k \) be an \((M, \varepsilon)\)-average of RIS of normalized special averages \((x_k)\), for \((n_k) \subset \{M + 3, M + 4, \ldots\}\) and \( 0 < \varepsilon < \theta_M \). Then \( \|x\| \leq D' \theta_M \), for some universal constant \( D' \) depending only on \( \theta \) and \( \theta_1 \).
Proof. Take a norming functional $f$ with a tree-analysis $(f_s)_{s \in T}$ such that $\|x\| = f(x)$. Using Fact 2.17 pick the subtree $T'$ satisfying (a) and (b) and the corresponding functional $f'$. Let $E$ be collection of all $\alpha \in T'$ maximal with respect to the property $\text{ord}(\alpha) \leq M - 1$. Notice that $E$ is $S_{M-1}$-allowable. 

Fix $\alpha \in E$. Then $\alpha$ is not terminal, so $f_\alpha = \theta_\alpha \sum_{s \in \text{succ}(\alpha)} f_s$. As in Remark 1.9 we partition $\text{succ}(\alpha) = \bigcup_{t \in A_\alpha} F_t$ in such a way that $(f_s)_{s \in F_t}$ is $S_{\text{ord}(s) - (M-1)}$-allowable for every $t \in A_\alpha$ and $(y_t)_{t \in A_\alpha}$ is $S_{M-1}$-allowable, where $y_t = \sum_{s \in F_t} f_s$. Let $A = \bigcup_{\alpha \in E} A_\alpha$ and notice that $(y_t)_{t \in A}$ is $S_{M-1}$-allowable. Let $H$ denote the set of all $k$ such that some $y_t$, $t \in A$, begins in $x_k$. Since $x$ is an $(M, \varepsilon)$-average we have $\|\sum_{k \in H} a_k x_k\| \leq \sum_{k \in H} a_k \leq 2\varepsilon$.

By definition of $H$ for any $\alpha \in E$ and $k \notin H$ with $f_\alpha(x_k) \neq 0$ there is an immediate successor of $\alpha$ beginning before $x_k$. Thus by (a) we have for any $k \notin H$

(c) for any $\alpha \in E$ with $f_\alpha(x_k) \neq 0$ the order of immediate successors of $\alpha$ is at most $n_k/4$,

(d) $\{y_t : t \in A, y_t(x_k) \neq 0\}$ restricted to supp$x_k$ is $S_1$-allowable.

Fix $k \notin H$ and $t \in A$ with $g_t(x_k) \neq 0$ and let $B^k_t = \{s \in F_t : f_s(x_k) \neq 0\}$.

Fix $s \in B^k_t$ and take the subtree $T_s$ of $T'$ consisting of $s$ (as a root) and of all successors of $s$ in $T'$. By Remark 1.9, using (b) and (c) we can add nodes in $T_s$ on level $n_k - \text{ord}(s)$ obtaining $(h_{s,r})_{r \in C_s}$, which is $S_{n_k-\text{ord}(s)}$-allowable satisfying 

$f_s(x_k) \leq \sum_{r \in C_s} \theta^{n_k-\text{ord}(s)} h_{s,r}(x_k)$.

Compute for $k \notin H$ using the above and (A)

$$f'(x_k) = \sum_{t \in A} \sum_{s \in B^k_t} t(s) f_s(x_k) \leq \theta_M \sum_{t \in A} \sum_{s \in B^k_t} \theta^{\text{ord}(s) - M} \sum_{r \in C_s} \theta^{n_k-\text{ord}(s)} h_{s,r}(x_k) \leq \theta_M \sum_{t \in A} \sum_{s \in B^k_t} \sum_{r \in C_s} \theta^{n_k-M} h_{s,r}(x_k).$$

Notice that the family $\{h_{s,r} : r \in C_s, s \in B^k_t\}$ for any fixed $t \in A, k \notin H$ is $S_{n_k-M+1}$-allowable. Therefore by (d) the family $\{h_{s,r} : r \in C_s, s \in B^k_t, t \in A\}$ for any fixed $k \notin H$ is $S_{n_k+2}$-allowable and hence since $x_k$ is a normalization of a $(n_k, \varepsilon_k)$-special average, we continue the estimation

$$\cdots \leq \theta_M \theta^{n_k-M} D^2 \theta^{-n_k+M-2} = D^2 \theta^{-2} \theta_M.$$

We compute

$$f(x) \leq f'(x) + 3\varepsilon \leq \sum_{k \notin H} a_k f'(x_k) + 5\varepsilon \leq D^2 \theta^{-2} \theta_M + 3\varepsilon \leq (D^2 \theta^{-2} + 5) \theta_M,$$

which ends the proof of Lemma.

2.4. Main results

Recall that a Banach space $(X, \|\cdot\|)$ is $\lambda$-distortable, $\lambda > 1$, if $X$ admits an equivalent norm $||\cdot||$, such that for any infinite dimensional subspace $Y$ of $X$ we have $\sup \{|x|/|y| : x, y \in Y, \|x\| = \|y\| = 1\} \geq \lambda$. A Banach space $X$ is arbitrarily distortable, if it is $\lambda$-distortable for any $\lambda > 1$. It is an open question if there exists a Banach space $\lambda$-distortable for some $\lambda > 1$, but not arbitrarily distortable. A natural candidate for such an example is the Tsirelson space $T[1/2]$.

Theorem 2.19. Let $X$ be a regular modified mixed Tsirelson space $T\{S_n, \theta_n\}$. If $\theta_n/\theta^n \nrightarrow 0$, then $X$ is arbitrarily distortable.
Proof. We prove that the norm given by \( \|x\|_n = \sup \{ \|E_i x\|, (E_i) \text{ S}_n\text{-admissible} \} \), \( x \in X \), \( c \theta^n/\theta_n \)-distorts \( X \), for any \( n \in \mathbb{N} \) and universal \( c > 0 \). Clearly \( \|x\| \leq \|x\|_n \leq 1/\theta_n \|x\| \). Fix an infinite dimensional subspace \( Y \) of \( X \). By Prop. 2.11 there is \( y \in Y \) with \( \|y\| \geq \theta_n /D \) and \( \|y\|_n \leq D \). On the other hand, again by Prop. 2.11 and Lemma 2.18 there is \( x \in Y \) with \( \|x\| \leq D^j \theta_n \) and \( \|y\| \geq 1 \). Considering \( x/\|x\| \) and \( y/\|y\| \) we obtain \( c \theta^n/\theta_n \)-distortion with \( c = 1/D^2 D^j \).

Recall that a Banach space \( X \) with a basis is called sequentially minimal [16], if any block subspace of \( X \) contains a block sequence \( (x_n) \) such that every block subspace of \( X \) contains a copy of a subsequence of \( (x_n) \). Notice that this property implies quasiminimality of \( X \).

**Theorem 2.20.** Let \( X \) be a regular modified mixed Tsirelson space \( T_M([\mathcal{S}_n, \theta_n]) \). If \( \theta_n/\theta^n \xrightarrow{n \to \infty} \), then \( X \) is sequentially minimal.

The theorem follows immediately from the following result:

**Lemma 2.21.** Let \( (x_k)_k \), \( (y_k)_k \) be RIS of Tsirelson \((2M_k - 3, \varepsilon_k)\)-averages, \( M_k > 4 \), \( \varepsilon < (6C)^{-1} \), with \( C \) as in Lemma 2.10, such that

1. \( x_k \) has an averaging tree \( (x^k_{i,j})_{i,j} \), \( (N^k_{i,j})_{i,j} \), \( (\varepsilon^k_{i,j})_{i,j} \), \( y_k \) has an averaging tree \( (y^k_{i,j})_{i,j} \), \( (N^k_{i,j})_{i,j} \), \( (\varepsilon^k_{i,j})_{i,j} \), both satisfying conditions (1) and (2) of Lemma 2.10 for any \( k \),
2. \( \text{minsupp } x^k_{i,j} = \text{minsupp } y^k_{i,j} \) and \( \|x^k_{i,j}\| = \|y^k_{i,j}\| = 1 \) for any \( k, i, j \),
3. \( \varepsilon_k \leq \varepsilon_{2M_k} \), \( \varepsilon_k \leq \varepsilon_{2M_k} \leq 3 \varepsilon /2^{k+2} \) for any \( k \).

Then \( (x_k/\|x_k\|)_k \) and \( (y_k/\|y_k\|)_k \) are equivalent.

Notice first that Lemma above yields Theorem 2.20, as given a block sequence \( (w_n) \) in \( X \), a block subspace \( Y \) of \( X \) and \( k \in \mathbb{N} \), we can choose block sequences \( (w_n) \subset [\{w_n\}] \) and \( (v_n) \subset Y \) satisfying the assertion of Lemma 2.14 for \( 2M_k - 3 \). Passing to a subsequences if necessary and using a small perturbations we obtain block sequences \( (u'_i) \) and \( (v'_i) \) of the form \( u_i' = u_i + \delta_i e_i, \ v_i' = v_i + \delta_i e_i \), for some \( (\delta_i) \subset \mathbb{N} \) with \( \delta_i = \text{minsupp } u_i' = \text{minsupp } v_i' \) for each \( i \) and small \( (\delta_i) \subset (0,1) \), which are equivalent to \( (u_i) \) and \( (v_i) \) respectively and satisfy the assertion of Lemma 2.14 for \( 2M_k - 3 \). Then by Fact 2.7 construct on these sequences two Tsirelson \((2M_k - 3, \varepsilon_k)\)-averages with averaging trees as in Lemma 2.21 with equal systems of weights, errors and coefficients, obtaining \( x_k \) and \( y_k \) for each \( k \in \mathbb{N} \).

Now we proceed to the proof of Lemma 2.21.

**Proof.** Notice first that by Lemmas 2.4 and definition of a Tsirelson average we have estimation

\[
\theta^{2M_k-3}/4 \leq \|x_k\| \leq 5\theta^{-2} \theta^{2M_k-3}, \quad k \in \mathbb{N},
\]

and the same estimation for \( \|y_k\|, k \in \mathbb{N} \).

We show first that \( (y_k/\|y_k\|)_k \) dominates \( (x_k/\|x_k\|)_k \). Let \( x = \sum_k d_k x_k/\|x_k\| \) be of norm 1, with \( (d_k) \subset [0,1] \), and take its norming functional \( f \) with a tree-analysis \( (f_n)_{n \in \mathbb{N}} \). Let \( y = \sum_k d_k y_k/\|y_k\| \). By Fact 2.17 we can assume with error \( \varepsilon \) that \( \text{ord}(\alpha) < M_k+1/4 \leq M_k+1 \) for any \( \alpha \in T \) with \( f(\alpha) \neq 0 \). For any \( k > 1 \) let

\[
\mathcal{E}_k = \{ \alpha \in T : f(\alpha) \text{ begins at } x_k \text{ and has a sibling beginning before } x_k \}.
\]

By our reduction \( \text{ord}(\alpha) < M_k-4 \) for any \( \alpha \in \mathcal{E}_k, k \geq 2 \). We replace in the tree-analysis of \( f \) each functional \( f_\alpha \), \( \alpha \in \mathcal{E}_k \), by two functionals \( g_\alpha = f_\alpha \text{ supp } x_k \) and \( k_\alpha = f_\alpha - g_\alpha \), obtaining a tree-analysis of a functional \( g \) on the space \( X_2 = T([\mathcal{S}_n A_2, \theta_n]) \), which by Lemma 1.7 is 3-isomorphic to \( X \).

Notice that \( \{g_\alpha \}_{\alpha \in \mathcal{E}_k, k \geq 2} \) have pairwise disjoint supports and \( \bigcup_{\alpha \in \mathcal{E}_k} \text{ supp } g_\alpha \cap \text{ supp } x_k = \text{ supp } f \cap \text{ supp } x_k \), hence \( f \text{ supp } x_k = \sum_{\alpha \in \mathcal{E}_k} \ell(\alpha) g_\alpha \). For each \( k \geq 2 \) consider the set \( J_k = \{ i : \text{ some } g_\alpha \text{ begins at } x_k^{M_k} \} \). Notice that by our reduction \( (g_\alpha)_{\alpha \in \mathcal{E}_k} \) is \( S_{M_k-4}\text{-allowable} \), thus \( (x_k^{M_k})_{i \in J_k \text{ min } J_k} \) is \( S_{M_k-4}\text{-admissible} \) and
recall that $x_k$ is an $(M_k - 3, \varepsilon)$-average of $(x_{k,i}^{M_k})$. Let $g'_\alpha, \alpha \in \mathcal{E}_k$, be the restriction of $g_\alpha$ to $\cup_{i \neq k} \text{supp} x_{k,i}^{M_k}$. Then we have the following estimation

$$f(x) = \frac{d_1}{\|x\|} f(x_1) + \sum_{k \geq 2} \frac{d_k}{\|x_k\|} f(x_k)$$

$$\leq \frac{d_1}{\|x\|} f(x_1) + \sum_{k \geq 2} \frac{d_k}{\|x_k\|} \sum_{\alpha \in \mathcal{E}_k} t(\alpha) g'_\alpha(x_k) + \sum_{k} \frac{d_k}{\|x_k\|} \|a_{k,k}^{M_k} \cdot x_{k,k}^{M_k}\|$$

$$\leq \frac{d_1}{\|x\|} f(x_1) + \sum_{k \geq 2} \frac{d_k}{\|x_k\|} \sum_{\alpha \in \mathcal{E}_k} t(\alpha) g'_\alpha(x_k) + 8 \varepsilon \theta^{-2M_k+3}$$

$$\leq \frac{d_1}{\|x\|} f(x_1) + \sum_{k \geq 2} \frac{d_k}{\|x_k\|} \sum_{\alpha \in \mathcal{E}_k} t(\alpha) g'_\alpha(x_k) + \varepsilon .$$

Fix $k \geq 2$. Notice that by definition the set $\{g'_\alpha : g'_\alpha(x_{k,i}^{M_k}) \neq 0\}$ restricted to the support of $x_{k,i}^{M_k}$ is $S_1$-allowable for any $i$. Therefore by Lemma 2.10 we pick suitable partition $(V_\alpha)_{\alpha \in \mathcal{E}_k}$ of nodes $(x_{k,i}^{0})_i$ with minsupp $x_{k,min, \alpha}^{0} \geq \text{minsupp} g'_\alpha$ for each $\alpha \in \mathcal{E}_k$ and by definition of a Tsirelson average obtain

$$\sum_{\alpha \in \mathcal{E}_k} t(\alpha) g'_\alpha(x_k) \leq C \sum_{\alpha \in \mathcal{E}_k} t(\alpha) \| \sum_{i \in V_\alpha} a_{k,i}^{0} \cdot \text{minsupp} x_{k,i}^{0} \| \|T[S_1, \theta]\| + C \varepsilon_k$$

$$\leq C \sum_{\alpha \in \mathcal{E}_k} t(\alpha) \| \sum_{i \in V_\alpha} a_{k,i}^{0} \cdot \text{minsupp} y_{k,i}^{0} \| \|T[S_1, \theta]\| + C \varepsilon_k$$

$$\leq 4C \sum_{\alpha \in \mathcal{E}_k} t(\alpha) \| \sum_{i \in V_\alpha} a_{k,i}^{0} \cdot y_{k,i}^{0} \| + C \varepsilon_k$$

$$\leq 4C \sum_{\alpha \in \mathcal{E}_k} t(\alpha) h_\alpha(y_k) + C \varepsilon_k ,$$

where $h_\alpha$ is a norming functional on $X$ with $h_\alpha(y_k) = \| \sum_{i \in \cup_{V_\alpha}} a_{k,i}^{0} \cdot y_{k,i}^{0} \|$ and supp $h_\alpha \subset \cup_{i \in V_\alpha} \text{supp} y_{k,i}^{0}$, thus minsupp $h_\alpha \geq \text{minsupp} x_{k,min, \alpha}^{0} \geq \text{minsupp} g'_\alpha$ for each $\alpha \in \mathcal{E}_k$. We modify the tree-analysis of $g$, replacing each node $g_\alpha$, $\alpha \in \mathcal{E}_k$, $k \geq 2$, by the functional $h_\alpha$. As minsupp $h_\alpha \geq \text{minsupp} g_\alpha$ for each $\alpha$, we obtain a tree-analysis of some norming functional $h$ on $X_2$. We compute, by Lemma 1.7 and above estimations including the estimation on the norms of $(x_k)_k$ and $(y_k)_k$,

$$1 = f(x) \leq d_1 + \sum_{k \geq 2} \frac{d_k}{\|x_k\|} \sum_{\alpha \in \mathcal{E}_k} t(\alpha) g_\alpha(x_k) + \varepsilon$$

$$\leq d_1 + 80C\theta^{-2} \sum_{k \geq 2} \frac{d_k}{\|y_k\|} \sum_{\alpha \in \mathcal{E}_k} t(\alpha) h_\alpha(y_k) + 4C \sum_{k \geq 2} \frac{\varepsilon_k}{\theta^{2M_k-3}} + \varepsilon$$

$$\leq d_1 + 80C\theta^{-2} h(\sum_{k \geq 2} \frac{d_k}{\|y_k\|} y_k) + 3C \varepsilon \leq 241C\theta^{-2}\|y\| + 1/2 ,$$

which means that $(y_k/\|y_k\|)_k$ dominates $(x_k/\|x_k\|)_k$. Since the conditions are symmetric, the opposite domination follows analogously.

3. Strictly singular non-compact operators

3.1. Spaces defined by families $(A_n)_n$

In spaces defined by families $(A_n)_n$ the crucial tool is formed by $\ell_p$-averages.
Definition 3.1. A vector \( x \in X \) is called a \( C-\ell_p \)-average of length \( m \), for \( p \in [1, \infty) \), \( m \in \mathbb{N} \) and \( C \geq 1 \) if \( x = \sum_{i=1}^{m} x_i / \| \sum_{i=1}^{m} x_i \| \) for some normalized block sequence \( (x_n)_{n=1}^{\infty} \) which is \( C \)-equivalent to the unit vector basis of \( l_p^m \).

Recall that an operator on a Banach space \( X \) is called strictly singular if its restriction to any infinite dimensional subspace of \( X \) is not an isomorphism.

Definition 3.2. [35] Let \( X \) be a Banach space with a basis \( (e_n) \). Then \( X \) is in

1. Class 1, if any normalized block sequence in \( X \) has a subsequence equivalent to a subsequence of \( (e_n) \).
2. Class 2, if each block sequence has further normalized block sequences \( (x_n) \) and \( (y_n) \) such that the map \( x_n \mapsto y_n \) extends to a bounded strictly singular operator between \( [(x_n)] \) and \( [(y_n)] \).

T. Schumpecht asked in [35] if any Banach space contains a subspace with a basis which is either of Class 1 or Class 2 and gave the existence of sufficient condition for the existence of strictly singular non-compact operator in the space.

Theorem 3.3. (Thm. 1.1 [35]) Let \( (x_n) \) and \( (y_n) \) be two normalized basic sequences generating spreading models \( (u_n) \) and \( (v_n) \) respectively. Assume that \( (u_n) \) is not equivalent to the u.v.b. of \( e_0 \) and \( (u_n) \) strongly dominates \( (v_n) \), i.e.

\[
\| \sum_{i=1}^{\infty} a_i v_i \| \leq \max_{n \in \mathbb{N}} \delta_n \max_{\# F \leq n} \| \sum_{i \in F} a_i u_i \|
\]

for some sequence \( (\delta_n) \) with \( \delta_n \to 0 \), \( n \to \infty \). Then the map \( x_n \mapsto y_n \) extends to a bounded strictly singular operator between \( [(x_n)] \) and \( [(y_n)] \).

Theorem 3.4. Let \( X = T[(A_n, \frac{\alpha}{n})] \) be a regular \( p \)-space, with \( p \in [1, \infty) \). Then

1. if \( \inf_n a_n > 0 \), then \( X \) is saturated with subspaces of Class 1.
2. if \( a_n \to 0 \), \( n \to \infty \), then \( X \) is not in Class 2.

Proof. (1). We show that any block subspace of \( X \) contains a normalized block sequence \( (u_n) \) with the following "blocking principle" ("lifting property" in [33]): any normalized block sequence \( (y_j) \) is equivalent to any \( (u_k) \), with \( y_j < u_{k+1} \) and \( u_k < y_{j+1} \). It follows that the subspace \( [(u_n)] \) is sequentially minimal.

By Prop. 2.10 [29] any block subspace of \( X \) contains an \( \ell_p \)-asymptotic subspace of \( X \). Let \( W \) be such \( \ell_p \)-asymptotic subspace, spanned by a normalized block sequence \( (w_k) \). Let \( C \) be the asymptotic constant of \( W \), i.e. any normalized block sequence \( (z_i)_{i=1}^{\infty} \) with \( z_i > n \) in \( W \) is \( C \)-equivalent to the u.v.b. of \( \ell_p^m \).

For any subspace \( Y \) of \( X \) spanned by normalized block sequence \( (y_n) \) let \( \| \sum_{i=1}^{\infty} a_i y_i \|_{Y, \infty} = \sup_{n \in \mathbb{N}} |a_n| \).

Fix two strictly increasing sequences of integers \( (m_n) \subset \mathbb{N} \) and \( (N_j) \subset \mathbb{N} \) and take normalized block sequences \( (v_n) \) of \( (u_k) \) and \( (w_j) \) of \( (v_n) \) such that

1. \( v_n > m_n \) in \( W \) for any \( n \),
2. for any \( y \in [(v_i),n] \) we have \( \| y \|_{W, \infty} < 1/(8m_n^5) \), for any \( n \),
3. \( w_j > N_j \) in \( V = [(v_n)] \) for any \( j \),
4. for any \( y \in [(u_i),j] \) we have \( \| y \|_{V, \infty} < 1/(8N_j^5) \), for any \( j \),
5. \( \sqrt{N_j} \geq C2^{j+7} \) for any \( j \)
6. \( N_j \theta m_n < 1/2^{n+5} \) for any \( n \geq j \) (in particular \( m_n \geq N_j \) for any \( n \geq j \))
7. \( \theta m_n \sum_{i<n} \# v_i < 1/2^{n+5} \) for any \( n \)

Notice that every vector \( y \in [(v_i),n] \) is an \( 2C-\ell_p \)-average of length \( m_n \) of some normalized block sequence \( (y_i)_{i=1}^{m_n} \) of \( (w_k) \). Indeed, by Claim 3.8 [29] and condition (2) split \( y \) into \( (F y)_i^{m_n} \) with almost equal norm, obtaining by condition (1) and \( \ell_p \)-asymptoticity of \( W \) that \( y \) is a suitable average. The same holds in \( V \): every vector \( y \in [(u_i),j] \) is an \( 2C-\ell_p \)-average of length \( N_j \) of some normalized block sequence \( (y_i)_{i=1}^{N_j} \) (block with respect to \( (v_n) \)).
We show that in such setting we can prove the above Theorem repeating the proof of Theorem 3.1 [29]. We consider any normalized block sequence \((y_j)\) of \((u_j)\) and as \((z_j)\) we take \((u_{k_j})\) with \(y_j < u_{k_j} < y_{j+1}\). By the above observation \(y_j = (y_1' + \cdots + y_N')/\|y_1' + \cdots + y_N'\|\) and \(u_{k_j} = (u_1' + \cdots + u_{N_j}')/\|u_1' + \cdots + u_{N_j}'\|\), where \((y_j')_{j=1}^N\) and \((u_{k_j}')_{j=1}^N\) are normalized block sequences with respect to \((v_j)\). Notice that \((N_j)\) are big enough by condition (5). We again use the above observation obtaining that each \(y_j'\) and \(v_j'\) is an \(\ell_p\)-average of a block sequence of \((w_k)\), of suitable length with parameters satisfying the assertion of a version of Lemma 3.2 [29] for \(C\)-averages instead of 2-averages (by conditions (6) and (7)). Therefore repeating the proof of Theorem 3.1 [29] we obtain uniform equivalence of \((y_j)\) and \((u_{k_j})\) and hence "blocking principle" stated above.

(2). Fix a block subspace \(Y\) of \(X\). By Theorem 2.9 [29] \(p\) is in Krivine set of \(Y\). Take finite normalized block sequences \((y_i)\) such that for some \((m_i) \subset \mathbb{N}\)

1. each \(y_i\) is 2 \(-\ell_p\)-averages of length \(N_i \geq (2m_i)^p\).
2. \(\theta_{m_i} \sum_{j<n} \# \text{supp } y_j \leq 1/2^{i+5}\) for any \(i\).
3. \(2^{i+5}\theta_{m_i} \to 0, \ i \to \infty\).

Passing to a subsequence we can assume that \((y_i)\) generates a spreading model \((v_i)\).

**Lemma 3.5.** The spreading model \((v_i)\) is strongly dominated by the u.h.b. of \(\ell_p\).

**Proof.** Take \(k \in \mathbb{N}\) and \((a_i)_{i=1}^N) \in c_0(0) with \(||(a_i)||_\infty = 1/k^2\) and \(||(a_i)||_p = 1\). Choose \(M\) by (3) in definition of \((y_i)\) with \(N\theta_{m_i} \leq 1/2^{i+5}\) for any \(i > M\) and \(1/2^M \leq 1/k\). We have \(||\sum_{i=1}^N a_i v_i|| \leq 2\||\sum_{i=1}^{N+M} a_i y_i||\), where \(a_i \Rightarrow a_i, i = 1, \ldots, N\).

Take a norming functional \(f\) with a tree-analysis \((f_t)_{t \in \tau}\) and \(f \subset \text{supp } y\), where \(y = \sum_{i=1}^{N+M} a_i y_i\). By Lemma 2.5 [29] up to multiplying by \(36\) we can assume that for any \(f_t\) and \(y_i\) we have either \(\text{supp } f_t \subset y_i\), \(\text{supp } f_t \supset \text{supp } y_i \cap \text{supp } f\) or \(\text{supp } f_t \cap \text{supp } y_i = \emptyset\). We say that \(f_t\) covers \(y_i\), if \(t\) is maximal in \(\tau\) with \(\text{supp } f_t \supset \text{supp } y_i \cap \text{supp } f\).

Let \(A = \{t \in \tau: f_t\text{ covers some } y_i\}\). Given any \(t \in A\) let \(I_t = \{i = 1 + M, \ldots, N + M: f_t\text{ covers } y_i\}\). Let \(\theta_{m_i}\) be the weight of \(f_t\). If \(m_i > m_i\) for some \(i \in I_t\) let \(i_i\) be the maximal element of \(I_t\) with this property. Otherwise let \(i_i = 0\).

First let \(I_t = \{i \notin I_t: \text{supp } y_i \cap \text{supp } f \subset \text{supp } f_t\}\). Notice that for any \(i \in I_t\) there is some \(f_t\)-successor of \(f_t\) so that \(\text{supp } y_i \cap \text{supp } f \subset \text{supp } f_t\). Hence

\[ f_t(\sum_{i \in L_t} \tilde{a}_i y_i) \leq \theta_{m_i} (\sum_{i \in I_t} f_t(\tilde{a}_i y_i)) \leq N\theta_{m_i} \leq 1/2^{i+2}\]

Thus \(f(\sum_{i \in A} y_i) \leq 1/2^M\) and we erase this part for all \(t\) with error \(1/k\). Notice that by condition (2) in choice of \((y_i)\) we have

\[ f_t(\sum_{i < i_t} y_i) \leq \theta_{m_i} \sum_{i < i_t} \# \text{supp } y_i \leq 1/2^{i+2}, \]

so we can again erase this part for all \(t\) with error \(1/k\).

Let \(g\) be the restriction of \(f\) to \(\cup_{t \in A} \text{supp } y_i\) and \(h = f - g\). First we consider \(g(y) = \sum_{t \in A} t(f_t)\tilde{a}_i f_t(y_i).\)

Let \(B = \{t \in A: \text{ord}(f_t) \leq k\}\), hence \(\#B \leq k\). Then \(\sum_{t \in B} \tilde{a}_i f_t(y_i) \leq \#B/k^2 \leq 1/k\), hence we can erase this part with error \(1/k\). Notice that \(\sum_{t \in A \setminus B} \text{ord}(f_t)/\text{ord}(f_t)^{1/q}\) is a norming functional on \(\ell_p\), hence

\[ \sum_{t \in A \setminus B} \tilde{a}_i t(f_t) f_t(y_i) \leq \sum_{t \in A \setminus B} \tilde{a}_i \frac{\text{ord}(f_t)}{\text{ord}(f_t)^{1/q}} \leq \max_{n \geq k} \max_{\|\tilde{a}_i\|_{A \setminus B}} \|\tilde{a}_i\|_p \leq \max_{n \geq k} c_n. \]
We consider \( h(y) = \sum_{t \in A} \sum_{i \in I, i > i_t} \hat{a}_i \sum_{s \in J_t} t(f_s) f_s(y_i) \). Let \( D = \{ s \in J_t, i \in I_t, i > i_t, t \in A : \text{ord}(f_s) \leq k \} \). Then
\[
\sum_{t \in A} \sum_{i \in I, i > i_t} \sum_{s \in J_t} \hat{a}_i f_s(y_i) \leq \sum_{t \in A} \sum_{i \in I, i > i_t} \hat{a}_i 8(\#J_t)^{1/q} \theta_t,
\]
and we again erase this part with error \( 1/k \). For any \( i \in I_t, i > i_t \) for some \( t \in A \) we let \( r_i = \text{ord}(f_t)m_i \) and compute, using Hölder inequality,
\[
\sum_{t \in A} \sum_{i \in I, i > i_t} \sum_{s \in J_t} \hat{a}_i t(f_s) f_s(y_i) \leq \sum_{t \in A} \sum_{i \in I, i > i_t} \hat{a}_i 8(\#J_t)^{1/q} \theta_t,
\]
\[
\leq 8 \max_{n \geq k} c_n \sum_{t \in A} \sum_{i \in I, i > i_t} \hat{a}_i \frac{(#J_t)^{1/q}}{r_i^{1/q}}.
\]

We put all the estimates together obtaining
\[
f(y) \leq 36(0 \max_{n \geq k} c_n + 4/k).
\]

Therefore we proved that \( \Delta_n = \sup \{ \sum_{i \in \mathbb{N}} a_i v_i : \sup_{i \in \mathbb{N}} |a_i| \leq \varepsilon, \sum_{i \in \mathbb{N}} \| a_i \| = 1 \} \) converges to zero, as \( \varepsilon \to 0 \). By Lemma 2.4 \cite{35} there are some \( (\delta_n)_n \subset (0, \infty) \) with \( \delta_n \searrow 0 \) such that for any \( (a_i)_i \in \ell_0 \)
\[
\| \sum_{i \in \mathbb{N}} a_i v_i \| \leq \max_{n \in \mathbb{N}} \delta_n \max_{#F \leq n} \| (a_i)_{i \in F} \|_{\ell_p},
\]
which ends the proof of Lemma.

We continue the proof of Theorem 3.4. By the proof of \cite[Thm 2.9]{29} \( p \) is in the Krivine set of \( Y \) in Lemberg sense \cite{24} i.e. for any \( n \) there is a normalized block sequence \( (x^{(n)}_i)_i \subset Y \) generating spreading model \( (u^{(n)}_i)_i \) such that \( (u^{(n)}_i)_i \) is \( 1 \)-equivalent to the u.v.b. of \( \ell_p \).

Pick \( (m_n)_n \) such that \( \delta_m \leq 1/4^n \). Apply \cite[Prop. 3.2]{4} to constants \( C_n = 2^n, n \in \mathbb{N} \) and normalized block sequences \( (x^{(m_n)}_i)_i \), generating spreading models \( (u^{(m_n)}_i)_i \). We obtain thus a seminormalized block sequence \( (x_i)_i \), generating spreading model \( (u_i)_i \) which \( C_n \) dominates \( (u^{(m_n)}_i)_i \), for any \( n \in \mathbb{N} \). By Lemma 3.5 we have
\[
\| \sum_{i \in \mathbb{N}} a_i v_i \| \leq \max_{n \in \mathbb{N}} \delta_n \max_{#F \leq n} \| (a_i)_{i \in F} \|_{\ell_p}.
\]

Notice that \( (u_i)_i \) is not equivalent to the u.v.b. of \( c_0 \), thus by Theorem 3.3 we finish the proof.

In \cite{19} the construction of non-compact strictly singular operators was based on \( c_0 \)-spreading model of higher order in the dual space. However this method does not follow straightforward in case of \( p \)-spaces, as the observation below shows. In \cite{23} it was shown that Schlimprecht space \( S = T(\{ (A_n, 2^{\#(A_n+1)}_n) \} \) introduced in \cite{34} contains a block sequence \( (y_k)_k \) generating an \( \ell_p \)-spreading model. We show that no biorthogonal sequence to \( (y_k)_k \) generates a \( c_0 \)-spreading model. An analogous example is constructed in \cite{8}: a mixed Tsirelson space defined by \((S_n)_n \) which admits a block sequence \( (z_k)_k \) generating an \( \ell_p \)-spreading model (cf. Def. 3.7), such that no biorthogonal sequence to \( (z_k)_k \) generates \( c_0 \)-spreading model.
Proposition 3.6. Consider the sequence \((y_k)\) generating an \(\ell_1\)-spreading model constructed in [23], \(y_k = \sum_{m=1}^{k} v_{k,m}, k \in \mathbb{N}\). Take any block sequence \((y_k^*) \subset S^*\) so that \(y_k^*(y_k) = \delta_{1,k}\). Then the sequence \((y_k^*)\) does not generate a \(c_0\)-spreading model.

Proof. We can assume that \(\text{supp} y_k^* = \text{supp} y_k, k \in \mathbb{N}\). Consider two cases:

CASE 1. There is \(m_0 \in \mathbb{N}\), \(\delta > 0\) and an infinite \(K \subset \mathbb{N}\) with \(|y_k^*|_{\ell_1} \geq \delta\) for any \(k \in K\).

Let \(z_k^*\) be the restriction of \(y_k^*\) to the support of \(\sum_{m=1}^{m_0} v_{k,m}\), \(k \in K\). Then \((z_k^*)_{k \in K}\) is a seminormalized block sequence in \(S^*\), majorized by \((y_k^*)_{k \in K}\). Since by the form of \((v_m,k)\) the length of \(\text{supp}(\sum_{m=1}^{m_0} v_{k,m})\) is constant, we can pick some subsequence \((z_k^*)_{k \in L}\) of \((z_k^*)_{k \in K}\) consisting of, up to controllable error, equally distributed vectors, i.e. for some finite sequence \((a_k)_{k \in L} \subset \mathbb{R}\) and \((n_k,i)_{k \in L}\) with \(n_k,i < n_{k+1,i}\) and \(n_{k,i} < n_{k+1,i}\) we have \(|z_k - \sum_{i \in L} a_i e_{n_k,i}| \leq 1/2^k\), \(k \in L\). As the u.v.b. in \(S\) is subymmetric, the same holds for \((z_k^*)_{k \in L}\), thus \((z_k^*)_{k \in L}\) is equivalent to spreading model generated by itself. It follows that \((y_k^*)\) cannot generate \(c_0\)-spreading model.

CASE 2. If the first case does not hold, pick increasing \((N_j) \subset \mathbb{N}\) so that \[
\left| y_{N_j}^* \left( \sum_{m=1}^{N_{j-1}} v_{N_j,m} \right) \right| \leq 1/2^j, \quad j \in \mathbb{N}.
\]

Consider the norm of vectors \(z_j^* = y_{N_1}^* + \cdots + y_{N_j}^*, \quad j \in \mathbb{N}\). Put

\[
x_{N_1} = y_{N_1}, \quad x_{N_j} = \sum_{m=N_{j-1}+1}^{N_j} v_{N_j,m}, \quad j > 1.
\]

By the choice of \((N_j)\), we have \(y_{N_j}^*(x_{N_j}) \geq 1 - 1/2^j\) for any \(j \in \mathbb{N}\).

We estimate the norm of \(x_j = x_{N_1} + \cdots + x_{N_j}, \quad j \in \mathbb{N}\). We can assume that \((N_j)\) was chosen to increase fast enough so that \((x_{N_j})\) is \(D\)-equivalent to the unit basis of \(S\) (see Remark 5, Lemma 2 [23]). Thus \(||x_j|| \leq D_j/f(j)\) for every \(j \in \mathbb{N}\). By the choice of \((N_j)\) and definition of \(x_{N_j}\), we have \(z_j^*(x_j) \geq j - 1\). Hence

\[
||z_j^*|| \geq z_j^*(x_j)/||x_j|| \geq f(j)(j-1)/D_j \geq f(j)/2D, \quad j \in \mathbb{N}.
\]

Notice that the same scheme works if we replace \(N_1, \ldots, N_j\) by any \(N_{n_1}, \ldots, N_{n_j}\) in definition of \(z_j\), hence no subsequence of \((y_k^*)\) can produce a \(c_0\)-spreading model.

3.2. Spaces defined by families \((S_n)\)

Regarding the existence of strictly singular operators from subspaces of mixed Tsirelson spaces we prove the following result, which is a "localization" of Schumpecht result in mixed Tsirelson spaces. First recall the definition of a higher order \(\ell_1\)-spreading models.

Definition 3.7. We say that a normalized basic sequence \((x_n)_{n \in \mathbb{N}}\) in a Banach space generates an \(C - \ell_{1}^\alpha\)-spreading model, \(\alpha < \omega_1, C \geq 1\), if for any \(F \in S\) the sequence \((x_n)_{n \in F}\) is \(C\)-equivalent to the u.v.b. of \(\ell_{1}^\# F\).

Notice that in case of \(\alpha = 1\) we obtain the classical \(\ell_1\)-spreading model. We recall that \([M], M \subset \mathbb{N}\), denotes the family of all infinite subsequences of \(M\), \([M]^<\) - the family of all finite subsequences of \(M\).

Theorem 3.8. Let \(X = T'[\{(S_n, \theta_n)\}]\) or \(T_M[\{(S_n, \theta_n)\}]\) be a regular (modified) mixed Tsirelson space. If \(X\) contains a block sequence \((y_n)\) generating an \(\ell_1^\alpha\)-spreading model then there are a subspace \(Y \subset \{[y_n]\}\) and a strictly singular non-compact operator \(T : Y \to X\).
We recall that in [25] it was proved that if a sequence \((\theta_n)\) satisfies \(\lim_m \sup_n \frac{\theta_m}{\theta_n} > 0\) then the regular mixed Tsirelson space \(X = T([S_n, \theta_n])\) is subsequentially minimal if and only if any block subspace of \(X\) admits an \(\ell_1^\omega\)-spreading model, if and only if any block subspace of \(X\) has Bourgain \(\ell_1\)-index greater than \(\omega^\omega\). These conditions hold in particular if \(\sup \theta^n = 1\) [27]. In [8, 22] analogs of these results were studied in the partly modified setting.

To prove the theorem we first define an index measuring the best constant of \(\ell_1^\omega\)-spreading models generated by subsequences of a given sequence. Let \(\vec{\alpha} := (x_n)_{n \in N}\) be a normalized block sequence in \(X\). Set

\[\delta_{\alpha}(\vec{\alpha}) = \sup \{\delta > 0 : \exists M \in [N] \text{ such that } (x_n)_{n \in M} \text{ generates } \delta - \ell_1^\omega\text{-spreading model}\}.\]

The following properties of \(\delta_{\alpha}(\vec{\alpha})\) follows readily from the definition.

1. \(\delta_{\alpha}(x_n)_{n \in N} = \delta_{\alpha}(x_n)_{n \geq n_0}\) for all \(n_0 \in N\).
2. \(\delta((x_n)_{n \in M}) \leq \delta_{\alpha}(x_n)_{n \in [N]}\) for all \(M \in [N]\).
3. \(\delta_{\alpha}(\vec{\alpha})\) is a non-increasing family.

By standard arguments we may stabilize \(\delta_{\alpha}(\vec{\alpha})\). Namely passing to a subsequence we may assume that \(\delta_{\alpha}(x_n)_{n \in N} = \delta_{\alpha}(x_n)_{n \in M}\) for every \(M \in [N]\).

By the reflexivity of the space \(X\) [9], Bourgain theorem yields that \(\delta_{\alpha}(\vec{\alpha}) > 0\) for countably many \(\alpha\)'s, enumerate them as \((\alpha_n)\). As \(X\) is an \(\ell_1\)-asymptotic space it follows that \(\delta_{\alpha}(\vec{\alpha}) > 0\) for all \(n \in N\).

Inductively we choose \(M_1 \supset M_2 \supset \ldots\) infinite subsets of \(N\) such that

\[\delta_{\alpha_n}(x_j)_{j \in M_n} = \delta_{\alpha_n}(x_j)_{j \in L} \forall L \in [M_n].\]

We define the family

\[F_n = \{A \in [N] : \exists x^* \in B_X \text{ with } x^*(x_i) > 2\delta_{\alpha_n}((x_j)_{j \in M_n}) \text{ for all } i \in A\}.\]

By Theorem 1.1 [18] there exists \(N \in [M_n]\) such that

either \(S_{\alpha_n} \cap [N] \subset F_n\) or \(F_n \cap [N] \subset S_{\alpha_n}\).

In the first case by 1-unconditionality of the basis it follows that \((x_n)_{n \in N}\) and hence \((x_k)_{k \in M_n}\) contains a subsequence which generates \(2\delta_{\alpha_n} - \ell_1^\omega\)-spreading model, a contradiction. Hence additionally by the above and [30] we may assume that \((M_n)_{n \in N}\) satisfies also the following

\[F_n(M_n) \subset S_{\alpha_n},\] (3.1)

\[S_{\alpha_n}^{-1} \cap \{F \subset N : \min F \geq \min M_n\} \subset S_{\alpha_n}^{-1}.\] (3.2)

Let \(M = (m_i)_i\), where \(m_i = \min M_i\), be a diagonal set and let \(\delta_{\alpha_n} = \delta_{\alpha_n}((x_{m_i}))\). Passing to a subsequence we may assume that \(\sum_{n \geq n_0} n\delta_{\alpha_n} < 0.25\). Let \(\|\sum a_i x_{m_i}\| = 1\) and let \(x^* \in B_X\) such that \(\sum a_i x^*(x_{m_i}) = 1\).

By the unconditionality we may assume that \(x^*(x_{m_i}) \geq 0\) for every \(i\). Let \(2\delta_{\alpha_n} = 1\) and \(F_k = \{i : x^*(x_{m_i}) \in (2\delta_{\alpha_k}, 2\delta_{\alpha_k - 1}]\}\) and \(F_1^k = F_k \cap \{1, \ldots, k - 1\}\), \(F_2^k = F_k \cap \{k, k + 1, \ldots\}\).

From (3.1), (3.2) we get \(F_k^2 \subset S_{\alpha_n} \cap \{F \subset N : \min F \geq k\} = G_k\). It follows

\[\|\sum a_i x_{m_i}\| = \sum a_i x^*(x_{m_i}) = \sum_{k=1}^{\infty} \sum_{i \in F_k^1} a_i x^*(x_{m_i})
\]

\[= \sum_{k=1}^{\infty} \left( \sum_{i \in F_k^1} a_i x^*(x_{m_i}) + \sum_{i \in F_k^2} a_i x^*(x_{m_i}) \right)
\]

\[\leq \sum_{k=2}^{\infty} 2\delta_{\alpha_{k-1}}(k-1) \max_i |a_i| + \sum_{k=1}^{\infty} 2\delta_{\alpha_{k-1}} \sum_{i \in F_k^2} |a_i|
\]

\[\leq 0.5\|\sum a_i x_{m_i}\| + \sum_{k=1}^{\infty} 2\delta_{\alpha_{k-1}} \sup F_{\bar{\mu}} \sum_{i \in F_{\bar{\mu}}} |a_i|.
\]

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Therefore by the above inequalities we have
\[ \| \sum_i a_i x_{m_i} \| \leq 4 \sum_{k=1}^{\infty} \delta_{\alpha_k-1} (\|(x_{m_i})_i\|) \sup_{F \in \mathcal{G}_k} \sum_i |a_i| \text{ for all } (a_i), \tag{3.3} \]
where \( \mathcal{G}_k = S_{\alpha_k} \cap \{ F \subset N : \min F \geq k \} \).

Proof of Theorem 3.8. Let \( \hat{c} = (c_n)_{n \in \mathbb{N}} \) be the basis of \( X \). Recall [7, 9] that for every \( j \in \mathbb{N} \) and every \( (n, \theta_n) \)-average \( \sum_{i \in F} a_i c_i \) (special convex combination) of the basis we have
\[ \theta_n \leq \| \sum_{i \in F} a_i c_i \| \leq 2 \theta_n, \]

It follows readily that \( \delta_n (\hat{c}) \in [\theta_n, 2\theta_n] \) and \( \delta_n (\hat{c}) = 0 \).

Let \( (y_n)_{n \in \mathbb{N}} \) be a normalized block sequence \( (y_n)_{n \in \mathbb{N}} \) generating \( \ell_1^\infty \)-spreading model, i.e. for some \( c \geq 1 \)
\[ \| \sum_i a_i y_i \| \geq c \sum_{i \in F} |a_i| \forall n \in \mathbb{N}, \quad F \in \mathcal{S}_n, \quad \min F \geq n \]
By the previous reasoning for \( (x_n)_{n \in \mathbb{N}} = (c_n)_{n \in \mathbb{N}} \) we pick a \( M = (m_i) \in [N] \) and a sequence \( \alpha_k \not\sim \omega \) such that \( \sum_{k} k \delta_{\alpha_k} (|c_{m_i}|) < \infty \) and (3.3) holds. Setting \( D = \sum_{k} \theta_{\alpha_k-1} \leq \sum_{k} k \delta_{\alpha_k-1} (|c_{m_i}|) \) we have
\[ \| \sum_i a_i e_{m_i} \| \leq 8 \sum_{k} \theta_{\alpha_k-1} \sup_{F \in \mathcal{G}_k} \sum_{i \in F} |a_i| \]
\[ \leq \frac{8D}{c} \sup_{k} |a_i| \sum_{i \in F} \|e_{m_i}\| \]
\[ \leq \frac{8D}{c} \sum_{i \in F} |a_i| \]
It follows that the operator extending the mapping \( y_n \to e_{m_n} \) factors through a \( c_0 \)-saturated space and hence is strictly singular and non-compact (as \( (y_n)_{n \in \mathbb{N}} \) is normalized).

3.3. Remarks and questions

As a corollary to Theorem 3.4, part (1), we obtain that the (non-modified) Tzafriri space \( Y \) has an \( \ell_2 \)-asymptotic subspace \( Z \) which satisfies a blocking principle in the sense of [14]. The only known spaces with a blocking principle so far were similar to \( T, T^* \) and their variations. The two major ingredients used in [14] for proving the minimality of \( T^* \) are the blocking principle and the saturation with \( \ell_2^\infty \)'s. It is shown in [21] that Tzafriri space \( Y \) contains uniformly \( \ell_2^\infty \)'s. It is not known whether \( Y \) is uniformly saturated with \( \ell_2^\infty \)'s. In the opposite direction, we do not know if \( Z \) contains a convexified Tsirelson space \( T^{(2)} \) (which is equivalent to its modified version).

Aside from the main topic of our paper, we want to finish with some observations about subsymmetric sequences (i.e. basic sequences equivalent to all its subsequences) in two concrete spaces and pose corresponding related questions about the richness of the set of subsymmetric sequences in a Banach space. In 1977 Altshuler [2] (cf. e.g. 26) constructed a Banach space with a symmetric basis which contains no \( \ell_p \) or \( c_0 \), and all its symmetric basic sequences are equivalent. In 1981 C. Read [32] constructed a space with, up to equivalence, precisely two symmetric bases. More precisely, Read proved that any symmetric basic sequence in his space CR is equivalent either to the u.v.b. of \( \ell_1 \) or to one of the two symmetric bases of CR. A careful look at the papers of Altshuler and Read shows that their proofs work similarly for the more general case of all subsymmetric basic sequences. This observation leads to the following questions:

**Question 1.** Does there exist a space in which all subsymmetric basic sequences are equivalent to one basis, and that basis is not symmetric?

We remark that Altshuler’s space has a natural subsymmetric version but we do not know if it satisfies the above property.

**Question 2.** Does there exist a space with exactly two subsymmetric bases, which are not symmetric?


