Complex interpolation and complementably minimal spaces

by

P. G. Casazza, N. J. Kalton, Denka Kutzarova and M. Mastyło

Abstract

We construct a class of super-reflexive complementably minimal spaces, and study uniformly convex distortions of the norm on Hilbert space by using methods of complex interpolation.

1. Introduction

A Banach space X is called *(complementably) minimal* if every infinite-dimensional closed subspace E contains a (complemented) subspace isomorphic to X. These notions were introduced by Pełczyński [17] and Rosenthal [19]. Any minimal space must be separable and it is classical that the spaces ℓ_p for $1 \leq p < \infty$ and c_0 are complementably minimal. The space T^* (the dual of Tsirelson space) provides another example of a minimal but not complementably minimal space ([2], [3]). Recently Schlumprecht ([20], [21]) constructed the first example of a complementably minimal space other than the classical spaces ℓ_p and c_0 and this was a launching point for a number of remarkable developments in Banach space theory ([8], [15], [16]).

The space constructed by Schlumprecht is reflexive (see [20] and Proposition 2 below) but fails to be super-reflexive since it contains ℓ_∞^n's uniformly. Our main aim in this note is to show how interpolation methods can be used to extend Schlumprecht's construction and thereby introduce a class of complementably minimal super-reflexive spaces. We also show that interpolation can be used to tighten the known results on distortions of the norm in Hilbert space ([16]); precisely one can require the distorted norm to satisfy good uniform convexity and uniform smoothness conditions.

Our arguments depend heavily on the ideas of complex interpolation of Banach spaces first introduced by Calderón in 1964 ([1]). For another application of such ideas to problems of this nature see Daher [5].

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2. Some remarks on minimal and complementably minimal spaces

Proposition 1. *Every minimal Banach space is isomorphic to a subspace of a minimal space with an unconditional basis.*

Proof. If X is minimal and contains an unconditional basic sequence then it is clear that X embeds into a minimal space with unconditional basis. We argue that X must contain an unconditional basic sequence. For otherwise by [7] X contains an hereditarily indecomposable subspace Y. Since Y is not isomorphic to any proper subspace of itself ([8] Corollary 19 and Theorem 21) this is a contradiction. □

Proposition 2. *(1) A minimal space is either reflexive or isomorphic to a subspace of c_0 or ℓ_1.*
(2) A complementably minimal space is either c_0, ℓ_1 or is reflexive.

Proof. (1) follows immediately from the preceding Proposition, since any space with an unconditional basis contains c_0 or ℓ_1 or is reflexive. (2) follows quickly from (1). □

We remark that it is unknown if every complementably minimal space is prime, or whether complementable minimality passes to dual spaces in general. Note that the minimal space T^* has a non-minimal dual ([3]).

3. Some classes of complementably minimal spaces

We let c_{00} be the space of all finitely non-zero sequences. If E_1 and E_2 are finite intervals of natural numbers we write $E_1 < E_2$ to mean $\max E_1 < \min E_2$. If $x \in c_{00}$ and E is a subset of $\mathbf{N}$ we write $Ex = x\chi_E$. We will also need the concept of a *block subspace*: this is a subspace of c_{00} generated by a sequence (u_n) whose supports E_n satisfy $E_1 < E_2 < \cdots$.

We will consider spaces X determined by lattice norms $\|.\|_X$ on c_{00}. We will then let X be the space of all sequences x so that $\|x\|_X = \sup \|(x_1, \ldots, x_n, 0, \ldots)\|_X < \infty$. We abbreviate $\|x\|_{\ell_p}$ to $\|x\|_p$ if $1 \le p \le \infty$. If X and Y are two such spaces and $0 < \theta < 1$ we define $X^{1-\theta}Y^\theta$ to be the space Z defined by $\|z\|_Z = \inf\{\max(\|x\|_X, \|y\|_Y) : |z| = |x|^{1-\theta}|y|^\theta\}$. When working over the complex scalars, if either X or Y is separable then Z coincides with the usual complex interpolation space $[X, Y]_\theta$ (see [1]). It will, however, be easily seen that our results apply also in the real case.

We let $\mathcal{G}$ be the class of increasing functions $f : [1, \infty) \to [1, \infty)$ so that:
(1) $f(1) = 1$ and $f(x) < x$ if $x > 1$.
(2) $x/f(x)$ is concave.
(3) f is submultiplicative, i.e. $f(xy) \le f(x)f(y)$.
Suppose further $1 \le p < r \le \infty$, and $f \in \mathcal{G}$. We define $\mathcal{X}(p, r; f)$ to be collection of all sequence spaces X so that:
(4) $\|x\|_r \le \|x\|_X \le \|x\|_p$ for all $x \in c_{00}$.

(5) X is p-convex and r-concave (with constants one).
(6) If $E_1 < E_2 < \cdots < E_n$ are intervals in $\mathbf{N}$ then

$$\|x\|_X \geq \frac{1}{f(n)^{1/p-1/r}} (\sum_{i=1}^n \|E_i x\|_X^p)^{1/p}.$$

Then $\mathcal{X}(p,r;f)$ is non-empty. Furthermore if $r < \infty$ then any $X \in \mathcal{X}(p,r;f)$ is separable by r-concavity. If $r = \infty$ the same conclusion can be obtained from the fact that $f(n) = o(n)$ as $n \to \infty$, since X cannot then contain c_0. When $p = 1$ and $r = \infty$ it is clear that there is a unique space, which may be constructed by an inductive procedure ([20], [8]) $S = S(f)$ (Schlumprecht f-space) satisfying a minimality condition: $\|x\|_S \leq \|x\|_X$ for $X \in \mathcal{X}(1,\infty;f)$. Furthermore as shown by Schlumprecht ([20]) in this space $\|\sum_{i=1}^n e_i\|_S = n/f(n)$ where e_i are the basis vectors.

Proposition 3. *For any $f \in \mathcal{G}$ and $1 \leq p < r \leq \infty$, the space $S_{p,r} = \ell_t^{1-\theta} S^\theta$ where $\theta = \frac{1}{p} - \frac{1}{r}$ and $t = (1-\theta)r$, is the unique space in $\mathcal{X}(p,q;f)$ satisfying $\|x\|_{S_{p,r}} \leq \|x\|_X$ for all $X \in \mathcal{X}(p,r;f)$. Furthermore $\|\sum_{i=1}^n e_i\|_{S_{p,r}} = n^{1/p} f(n)^{1/r-1/p}$.*

Proof. For the case $r = \infty$ this follows by elementary convexification from the case $p = 1$, since the space $S_{p,\infty}$ coincides with the p-convexification of S. If $r < \infty$ it will follow easily by convexification or concavification from the case when $p < 2$ and $r = q$ the conjugate index of p. We therefore suppose $r = q$. First we prove that the space $S_{p,q}$ is in the class $\mathcal{X}(p,q;f)$. Indeed the only property to be verified is (3), and this is standard. If $0 \leq x \in c_{00}$ and $E_1 < E_2 < \cdots < E_n$ then we can write $x = u^{1-\theta} v^\theta$ where $0 \leq u, v \in c_{00}$ and $\|u\|_2 = \|v\|_S = \|x\|_{S_{p,q}}$. Thus

$$\begin{aligned}
\left(\sum_{j=1}^n \|E_j x\|_{S_{p,q}}^p \right)^{1/p} &\leq \left(\sum_{j=1}^n \|E_j u\|_2^{p(1-\theta)} \|E_j v\|_S^{p\theta} \right)^{1/p} \\
&\leq \left(\sum_{j=1}^n \|E_j u\|_2^2 \right)^{1/q} \left(\sum_{j=1}^n \|E_j v\|_S \right)^{1/p-1/q} \\
&\leq f(n)^{1/p-1/q} \|u\|_2^{2/q} \|v\|_S^{1/p-1/q} = f(n)^{1/p-1/q} \|x\|_{S_{p,q}}.
\end{aligned}$$

Conversely suppose $X \in \mathcal{X}(p,q;f)$. Then by Pisier's extrapolation theorem ([18] or [9]) there is a sequence space Y so that $X = Y^\theta \ell_2^{1-\theta}$ where $\theta = \frac{1}{p} - \frac{1}{q} = 2(\frac{1}{p} - \frac{1}{2})$. We show that $Y \in \mathcal{X}(1,\infty;f)$. Clearly $\|y\|_\infty \leq \|y\|_Y \leq \|y\|_1$ for all $y \in c_{00}$. Now suppose $0 \leq y \in c_{00}$ and $E_1 < E_2 < \cdots < E_n$ are disjoint intervals; let $y_i = E_i y$. Pick $y_i^* \in c_{00}$ supported on E_i with $\|y_i^*\|_{Y^*} = 1$ and $\langle y_i, y_i^* \rangle = \|y_i\|_Y$. Let $v_i = y_i^{\frac{1+\theta}{2}} (y_i^*)^{\frac{1-\theta}{2}} = y_i^\theta w_i^{1-\theta}$ where $w_i = (y_i y_i^*)^{1/2}$. Then $\|w_i\|_2 = \|y_i\|_Y^{1/2}$.

Now $X = Y^{\frac{1+\theta}{2}} (Y^*)^{\frac{1-\theta}{2}}$ (this follows for example from Lozanovskii's theorem ([12]) and the Re-iteration theorem ([1] and [4])). Also by the duality theorem

$X^* = Y^{\frac{1-\theta}{2}}(Y^*)^{\frac{1+\theta}{2}}$. Hence $\|v_i^*\|_{X^*} \le \|y_i\|_Y^{\frac{1-\theta}{2}}$ where $v_i^* = y_i^{\frac{1-\theta}{2}}(y_i^*)^{\frac{1+\theta}{2}}$. Hence $\|v_i\|_X \ge \|y_i\|_Y^{\frac{1+\theta}{2}}$.

We conclude that

$$\left\|\sum_{i=1}^n v_i\right\|_X \ge \frac{1}{f(n)^\theta}\left(\sum_{i=1}^n \|y_i\|_Y^{p(1+\theta)/2}\right)^{1/p}.$$

This implies

$$\left\|\sum_{i=1}^n v_i\right\|_X \ge \frac{1}{f(n)^\theta}\left(\sum_{i=1}^n \|y_i\|_Y\right)^{(1+\theta)/2}.$$

On the other hand

$$\left\|\sum_{i=1}^n v_i\right\|_X \le \|y\|_Y^\theta \left\|\sum w_i\right\|_2^{1-\theta}$$

or

$$\left\|\sum_{i=1}^n v_i\right\|_X \le \|y\|_Y^\theta \left(\sum_{i=1}^n \|y_i\|_Y\right)^{(1-\theta)/2}.$$

Combining these inequalities gives that

$$\|y\|_Y \ge \frac{1}{f(n)}\sum_{i=1}^n \|y_i\|_Y$$

so that $Y \in \mathcal{X}(1,\infty;f)$. Thus $Y \subset S$ with norm-one inclusion and similarly $Y^\theta \ell_2^{1-\theta} = X \subset S_{p,q}$ with norm one inclusion either by interpolation or simple calculation of norms. Finally we note that

$$\left\|\sum_{i=1}^n e_i\right\|_{S_{p,r}} \le \left\|\sum_{i=1}^n e_i\right\|_2^{1-\theta}\left\|\sum_{i=1}^n e_i\right\|_S^\theta = n^{1/p}f(n)^{1/p-1/q}.$$

The other inequality follows from (6) immediately since the basis vectors have norm one. ∎

Remark. The last condition shows that $S_{p,r}$ does not coincide with ℓ_p as a sequence space. It further follows that since the basis (e_n) of $S_{p,r}$ is subsymmetric that it has no subsequence equivalent to the ℓ_p-basis and therefore $S_{p,r}$ cannot be even isomorphic to ℓ_p.

The following remarkable result is due to Schlumprecht [21]:

Theorem 4. *If $f(x) = \log_2(x+1)$ then $S = S(f)$ is complementably minimal.*

We now prove a simple extension of a technique used by both Schlumprecht (see [20] and [21]) and Gowers and Maurey ([8]).

Proposition 5. *Suppose $f \in \mathcal{G}$ and that $\lim_{x\to\infty} f(x)x^{-a} = 0$ if $a > 0$. Suppose $1 \le p < \infty$, and suppose $\frac{1}{p} + \frac{1}{q} = 1$ (with appropriate interpration when $p = 1$). Suppose $X \in \mathcal{X}(p, \infty; f)$ Then:*
(1) If $n \in \mathbf{N}$ and $\epsilon > 0$ and W is a block subspace of c_{00} there is a block basic sequence $(u_1, u_2, \ldots, u_n)$ in W so that $\|u_i\|_X = 1$ for $1 \le i \le n$ and $\|u_1+\cdots+u_n\|_X \ge n^{1/p} - \epsilon$.
(2) If $n \in \mathbf{N}$ and $\epsilon > 0$ and W is a block subspace of c_{00} there is a block basic sequence $(u_1^*, u_2^*, \ldots, u_n^*)$ in W so that $\|u_i^*\|_{X^*} = 1$ for $1 \le i \le n$ and $\|u_1^* + \cdots + u_n^*\|_{X^*} \le n^{1/q} + \epsilon$.

Proof. (1) can be obtained immediately from Lemma 3 of Gowers-Maurey ([8]) by convexification. The proof of (2) is similar. Let $\beta_n = \beta_n(W)$ be the least constant so that for every k, ϵ there exists a normalized block basic sequence $(u_1^*, u_2^*, \ldots, u_n^*)$ in W with $k <$ supp u_1^* and $\|\sum_{i=1}^n u_i^*\|_{X^*} \le \beta_n + \epsilon$. Then it is easy to see that $\beta_{mn} \ge \beta_m\beta_n$ for m, n. Also by q-concavity of X^* we have $\beta_n \ge n^{1/q}$ for all n. On the other hand one can verify easily by duality that for all block basic sequences $(u_1^*, \ldots, u_n^*)$ we have $\|\sum_{i=1}^n u_i^*\|_{X^*} \le f(n) \left(\sum_{i=1}^n \|u_i^*\|_{X^*}^q\right)^{1/q}$ and so $\beta_n \le f(n)n^{1/q}$. Hence $\beta_n \le (\beta_{n^k})^{1/k} \le (f(n^k))^{1/k} n^{1/q}$ and so $\beta_n = n^{1/q}$. □

The key to Schlumprecht's argument for Theorem 4 is the following, which combines his Lemma 2 and Theorem 3. (Note that if $(u_i)_{i=1}^n$ is a normalized block basic sequence with $\|u_1 + \cdots + u_n\| > n - \epsilon$ then $(u_i)_{i=1}^n$ is $(1-\epsilon)^{-1}$ equivalent to the ℓ_1^n basis.)

Proposition 6. (Schlumprecht [21]) *Suppose $f(x) = \log_2(x + 1)$ and that $(u_n)_{n=1}^\infty$ is a normalized block basic sequence in $S = S(f)$. Let $v_n = 2^{-n} \sum_{i=2^n+1}^{2^{n+1}} u_i$ and suppose $\lim_{n\to\infty} 2^n(1 - \|v_n\|_S) = 0$. Then (v_n) has a subsequence (w_n) so that (w_n) is equivalent to the unit vector basis of $S(f)$.*

Remark. The conclusions of Propositions 5 and 6 are all we require for our main result. Thus Theorem 8 below will hold for any $f \in \mathcal{F}$ for which these Propositions hold; clearly this is a much wider class than just the singleton $f(x) = \log_2(x + 1)$ but it has not been precisely determined to date.

Proposition 7. *Suppose $f(x) = \log_2(x + 1)$ and $1 \le p < r \le \infty$. Suppose (u_n) is a normalized block basic sequence in $S_{p,r} = S_{p,r}(f)$. Let $v_n = 2^{-n/p} \sum_{i=2^n+1}^{2^{n+1}} u_i$ and suppose that $\lim_{n\to\infty} 2^n(1 - \|v_n\|_{S_{p,r}}) = 0$. Then (v_n) has a subsequence (w_n) which is equivalent to the unit vector basis of $S_{p,r}$.*

Proof. If $r = \infty$ this follows immediately from the fact that $S_{p,\infty}$ is the p-convexification of S. If $r < \infty$ we can assume each $u_i \ge 0$. Let $\epsilon_n = 2^n(1 - \|v_n\|_{S_{p,r}})$, so that $\lim_{n\to\infty} \epsilon_n = 0$. Note that each $u_i = x_i^\theta y_i^{1-\theta}$ where (x_i) is a normalized positive block basic sequence in S and (y_i) is a normalized block basic sequence in ℓ_t as in Proposition 3. Then

$$\|v_n\|_{S_{p,r}} \leq 2^{-n/p} \left\| \sum_{i=2^n+1}^{2^{n+1}} x_i \right\|_S^\theta \left\| \sum_{i=2^n+1}^{2^{n+1}} y_i \right\|_t^{1-\theta}$$

so that

$$1 - 2^{-n}\epsilon_n \leq 2^{n((1-\theta)/t-1/p)} \left\| \sum_{i=2^n+1}^{2^{n+1}} x_i \right\|_S^\theta$$

and this simplifies to

$$\left\| \sum_{i=2^n+1}^{2^{n+1}} x_i \right\|_S \geq (2^{n\theta} - 2^{n(\theta-1)}\epsilon_n)^{1/\theta} \geq 2^n - \theta^{-1}\epsilon_n.$$

It thus follows from Proposition 6 that a suitable subsequence $(x_{k(n)})$ is equivalent to the unit vector basis of S and so for some constant K and any finitely nonzero sequence (d_n) we have

$$\left\| \sum_{n=1}^{\infty} d_n x_{k(n)} \right\|_S \leq K\|d\|_S.$$

Now if $0 \leq a, b \in c_{00}$ then

$$\left\| \sum_{n=1}^{\infty} a_n^\theta b_n^{1-\theta} v_{k(n)} \right\|_{S_{p,r}} \leq \left\| \sum a_n x_{k(n)} \right\|_S^\theta \left\| \sum b_n y_{k(n)} \right\|_t^{1-\theta}$$

Hence

$$\left\| \sum_{n=1}^{\infty} a_n^\theta b_n^{1-\theta} v_{k(n)} \right\|_{S_{p,r}} \leq K^\theta \|a\|_S^\theta \|b\|_t^{1-\theta}$$

and so for any $d \in c_{00}$

$$\left\| \sum_{n=1}^{\infty} d_n v_{k(n)} \right\|_{S_{p,r}} \leq K^\theta \|d\|_{S_{p,r}}.$$

However the minimality property of the norm on $S_{p,r}$ clearly implies that

$$\|d\|_{S_{p,r}} \leq \left\| \sum_{n=1}^{\infty} d_n v_{k(n)} \|v_{k(n)}\right\|_{S_{p,r}}^{-1} \|_{S_{p,r}} \leq C_0 \left\| \sum_{n=1}^{\infty} d_n v_{k(n)} \right\|_{S_{p,r}},$$

where $C_0 = \sup_n \|v_{k(n)}\|_{S_{p,r}}^{-1}$. Taking $w_n = v_{k(n)}$ the result is proved. □

Theorem 8. *Suppose $f(x) = \log_2(x+1)$ and that $1 \leq p < r \leq \infty$. Then the spaces $S_{p,r}(f)$ and $S_{p,r}(f)^*$ are complementably minimal.*

Proof. By combining Proposition 5 and Proposition 7 we see that any block subspace of $S_{p,r}$ contains a normalized block basic sequence (w_n) equivalent to the unit vector basis of $S_{p,r}$. Let (g_n) be any bounded block basic sequence in $S^*_{p,r}$ so that $E_n = \operatorname{supp} g_n$ does not meet $\operatorname{supp} w_m$ when $m \neq n$ and $\langle w_n, g_n\rangle = 1$. Then we claim that $Px = \sum_{n=1}^{\infty}\langle x, g_n\rangle w_n$ defines a projection onto $[w_n]$. In fact if $x \in c_{00}$ then for a suitable constant C, and letting $M = \sup \|g_n\|_{S^*_{p,r}}$,

$$\begin{aligned}\|Px\|_{S_{p,r}} &\leq C\|(\langle x, g_n\rangle)\|_{S_{p,r}}\\ &\leq CM\|(\|E_n x\|)\|_{S_{p,r}}\\ &\leq CM\|x\|_{S_{p,r}}\end{aligned}$$

and so $[w_n]$ is complemented. This shows that $S_{p,r}$ is complementably minimal.

In $S^*_{p,r}$ we argue that any block subspace contains a normalized block basic sequence (u^*_n) so that $\left\|2^{-n/q}\sum_{i=2^n+1}^{2^{n+1}} u^*_i\right\|_{S^*_{p,r}} \leq \left(1+\frac{1}{n}2^{-n}\right)$ for all n. Choose a normalized block basic sequence (u_n) in $S_{p,r}$ so that $\operatorname{supp} u_n$ is contained in $\operatorname{supp} u^*_n$ and $\langle u_n, u^*_n\rangle = 1$. Then $\left\|2^{-n/p}\sum_{i=2^n+1}^{2^{n+1}} u_i\right\| \geq 1 - \frac{1}{n}2^{-n}$. It follows by the argument of the first part that we can select a sequence $k(n) \to \infty$ so that if $w_n = 2^{-k(n)/p}\sum_{i=2^{k(n)}+1}^{2^{k(n)+1}} u_i$ then (w_n) is equivalent to the unit vector basis of $S_{p,r}$ and complemented by the projection $Px = \sum_{n=1}^{\infty}\langle x, w^*_n\rangle w_n$ where $w^*_n = 2^{-k(n)/q}\sum_{i=2^{k(n)}+1}^{2^{k(n)+1}} u^*_i$. Thus $[w^*_n]$ is a complemented subspace of $S^*_{p,r}$ isomorphic to $S^*_{p,r}$. □

Remark. Of course in the cases when $1 < p < r < \infty$ the space $S_{p,r}$ is super-reflexive (in fact the given norm is uniformly convex). Note also that it is trivial that none of these spaces can contain a copy of c_0 or ℓ_p for any $1 \leq p < \infty$. The first such example was due to Tsirelson ([22]) and the first super-reflexive example of such a space was given by Figiel and Johnson ([6]). Note also that $S^* = S^*_{1,\infty}$ is complementably minimal. We remark also that the spaces $S_{p,r}$ for $1 < p < r < \infty$ are arbitrarily distortable in view of their minimality and a recent result of Maurey [13] (see also [14]). □

4. Distortions of Hilbert space

The distortion problem for Hilbert spaces was recently solved by Odell and Schlumprecht ([15], [16]); for the latest developments see Milman and Tomczak-Jaegermann [14] and Maurey [13]. In this brief section we show the following, a strengthening of Theorem 1.2 of [16].

Theorem 9. *Given $1 < p < 2$ (with conjugate index q) there is a constant $K = K(p)$ so that for any constant C there is a Banach space X isomorphic to ℓ_2 so that:*
(1) X contains no C-unconditional basic sequence.

(2) The modulus of convexity δ_X satisfies $\delta_X(\epsilon) \geq K^{-1}\epsilon^q$ for $0 \leq \epsilon \leq 1$.
(3) The modulus of smoothness ρ_X satisfies $\rho_X(\tau) \leq K\tau^p$ for $0 \leq \tau \leq 1$.

Proof. In fact this is an easy modification of Gowers-Maurey [8] Theorem 2, using the Odell-Schlumprecht theorem on the existence of asymptotic biorthogonal systems in ℓ_2 ([15], [16]). We shall suppose that all spaces are complex in the following argument. It is clear that the construction actually yields a space in which the underlying real space has no good unconditional basic sequences, so that the complex structure can be "forgotten" at the end.

Fix $\theta = 2/q$ and choose r an integer so that $r^{1-\theta} > 8C$. Using the notation of [8] let (A_n, A_n^*) be an asymptotic biorthogonal system with constant at most r^{-2}. For each n let Z_n^* be a countable dense subset of A_n^* and let $Z^* = \cup Z_n^*$. Let σ be an injection from the collections of all finite sequences in Z^* to $\mathbf{N}$. Let Γ_r be the collection of all $z^* \in \ell_2^*$ of the form $z^* = \sum_{i=1}^r z_i^*$ where $z_1^* \in Z_1^*$ and $z_{i+1}^* \in Z_{\sigma(z_1^*,\ldots,z_i^*)}^*$.

Define a new equivalent norm $\|.\|_Y$ on ℓ_2 by $\|x\|_Y = \max(\|x\|, r\max\{|\langle x, z^*\rangle| : z^* \in \Gamma_r\})$. Let $Y = (\ell_2, \|.\|_Y)$ and form the complex interpolation space $X = [Y, \ell_2]_\theta$.

Note first that X is $\theta-$Hilbertian ([18]) and so satifies the inequalities

$$\frac{1}{2}(\|x+y\|_X^p + \|x-y\|_X^p) \leq \|x\|_X^p + \|y\|_X^p$$

and

$$\frac{1}{2}(\|x+y\|_X^q + \|x-y\|_X^q) \geq \|x\|_X^q + \|y\|_X^q$$

From this it follows easily that one has the appropriate estimates on the moduli of convexity and smoothness.

It remains to show that X has no C-unconditional basic sequence. To see this let E be any block subspace of X. Arguing as in [8] one constructs a block basis $(z_1, \ldots, z_r)$ normalized in ℓ_2 and $z^* \in \Gamma_r$ so that $|\langle \sum_{i=1}^r z_i, z^*\rangle| \geq r-1$ and $\|\sum_{i=1}^r (-1)^i z_i\|_Y \leq 4r$.

Now $\|z^*\|_{Y^*} \leq r^{-1}$ and $\|z^*\|_2 \leq r^{1/2}$. Hence, by the duality theorem $\|z^*\|_{X^*} \leq r^{(3\theta-2)/2}$, whence it follows that $\|\sum_{i=1}^r z_i\|_X \geq \frac{1}{2}r^{(4-3\theta)/2}$. On the other hand $\|\sum_{i=1}^r (-1)^i z_i\|_2 \leq r^{1/2}$ so that $\|\sum_{i=1}^r (-1)^i z_i\|_X \leq 4^{1-\theta} r^{(2-\theta)/2}$. It follows no basic sequence in X has unconditional basis constant better than $\frac{1}{8}r^{1-\theta} > C$. □

Remark. In this theorem one cannot achieve a similar result with $p = q = 2$. For in this case it follows that X has bounded type two and cotype two constants (cf. [11] p. 77) and so by Kwapień's theorem ([10]) has bounded distance to Hilbert space.

References

[1] A. P. Calderón. Intermediate spaces and interpolation, the complex method. *Studia Math.* **24** (1964), 113–190.

[2] P. G. Casazza, W. B. Johnson and L. Tzafriri. On Tsirelson's space. *Israel J. Math.* **47** (1974), 191–218.

[3] P. G. Casazza and T. J. Shura. Tsirelson's space. Springer Lecture Notes vol. **1363** Springer, New York, 1988.

[4] M. Cwikel. Complex interpolation spaces, a discrete definition and reiteration. *Indiana Univ. Math. J.* **27** (1978), 1005–1009.

[5] M. Daher. Homéomorphismes uniformes entre les sphères unité des espaces d'interpolation. *C. R. Acad. Sci. (Paris)* **316** (1993), 1051–1054.

[6] T. Figiel and W. B. Johnson. A uniformly convex space which contains no ℓ_p. *Comp. Math.* **29** (1974), 179–190.

[7] W. T. Gowers. A new dichotomy for Banach spaces, preprint. ????

[8] W. T. Gowers and B. Maurey. The unconditional basic sequence problem. *J. Amer. Math. Soc.* **6** (1993), 851–874.

[9] N. J. Kalton. Differentials of complex interpolation processes for Köthe function spaces. *Trans. Amer. Math. Soc.* **333** (1992), 479–529.

[10] S. Kwapień. Isomorphic charcaterization of inner-product spaces by orthogonal series with vector-valued coefficients. *Studia Math.* **44** (1972), 583–595.

[11] J. Lindenstrauss and L. Tzafriri. Classical Banach spaces II, Function spaces. Springer, Berlin, 1979.

[12] G. Y. Lozanovskii. On some Banach lattices. *Siberian Math. J.* **10** (1969), 419–430.

[13] B. Maurey. A remark about distortion, preprint. ????

[14] V. D. Milman and N. Tomczak-Jaegermann. Asymptotic ℓ_p-spaces and bounded distortions. *Contemp. Math.* **144** (1993), 173–195.

[15] E. Odell and T. Schlumprecht. The distortion of Hilbert space. *Geom. and Funct. Anal.* **3** (1993), 201–217.

[16] E. Odell and T. Schlumprecht. The distortion problem, preprint. ????

[17] A. Pełczyński. Projections in certain Banach spaces. *Studia Math.* **19** (1960), 209–228.

[18] G. Pisier. Some applications of the complex interpolation method to Banach lattices. *J. d'Analyse Math.* **35** (1979), 264–281.

[19] H. P. Rosenthal. On a theorem of Krivine concerning block finite representability of ℓ_p in general Banach spaces. *J. Functional Analysis* **28** (1978), 197–225.

[20] T. Schlumprecht. An arbitrarily distortable Banach space. *Israel J. Math.* **76** (1991), 81–95.

[21] T. Schlumprecht. A complementably minimal space not containing c_0 or ℓ_p, preprint. ???

[22] B. S. Tsirelson. Not every Banach space contains an embedding of ℓ_p or c_0. *Funct. Anal. Appl.* **8** (1974), 138–141.