Remarks about Schlumprecht space

by

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Abstract

Let $\mathbf{S}$ denote the Schlumprecht space. We prove that

1. ℓ_∞ is finitely disjointly respresentable in $\mathbf{S}$;
2. $\mathbf{S}$ contains an ℓ_1-spreading model;
3. for any sequence (n_k) of natural numbers, $\mathbf{S}$ is isomorphic to the space $\left(\sum_{k=1}^{\infty} \oplus \ell_\infty^{n_k}\right)_{\mathbf{S}}$.

Let $(e_i)_{i=1}^{\infty}$ be the standard basis of the linear space c_{00}, the set of all finitely supported sequences. For $x = \sum_{i=1}^{\infty} a_i e_i \in c_{00}$, $\operatorname{supp} x$ denotes the set $\{i \in \mathbb{N} : a_i \neq 0\}$. A subset E of $\mathbb{N}$ is said to be an *interval* if there exist a, b such that

$$E = \{c \in \mathbb{N} : a < c < b\}.$$

For finite subsets E, F of $\mathbb{N}$, $E < F$ means $\max E < \min F$ or E is an empty set. For $x = \sum_{i=1}^{\infty} a_i e_i$ and a subset E of $\mathbb{N}$, Ex denotes the vector $Ex = \sum_{i \in E} a_i e_i$. Let $f : [1, \infty) \to [1, \infty)$ be the function defined by $f(x) = \log_2(x+1)$. The Schlumprecht space $\mathbf{S} = (\mathbf{S}, \|\cdot\|)$ is the completion of c_{00} with respect to the norm $\|\cdot\|$ which satisfies the following implicit equation:

$$\|x\| = \max\left\{ \|x\|_\infty, \sup_{E_1 < E_2 < \cdots < E_r} \frac{1}{f(r)} \sum_{i=1}^{r} \|E_i x\| \right\}. \tag{1}$$

Clearly, (e_i) is a 1-unconditional subsymmetric basis of $\mathbf{S}$. It is easy to see that every normalized block basis (y_i) of (e_i) dominates (e_i), i.e.

$$\left\| \sum_{i=1}^{\infty} a_i y_i \right\| \geq \left\| \sum_{i=1}^{\infty} a_i e_i \right\|$$

for all $(a_i) \in c_{00}$.

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Recall a sequence (x_n) in $\mathbf{S}$ is said to be *successive* if $\operatorname{supp} x_1 < \operatorname{supp} x_2 < \cdots$. A sequence of finite subspaces (V_n) is said to be *successive* if $x_j \in V_j \setminus \{0\}$, $x_k \in V_k \setminus \{0\}$ and $j < k$ implies that $\operatorname{supp} x_j < \operatorname{supp} x_k$.

In [15], Th. Schlumprecht introduced the space $\mathbf{S}$ and he showed that $\mathbf{S}$ is arbitrarily distortable. He also showed that $\mathbf{S}$ has neither ℓ_p-spreading model for any $1 < p < \infty$ nor c_0-spreading model. He asked the following question.

Question 1. *Does* $\mathbf{S}$ *contain an* ℓ_1*-spreading model?*

Recall that a vector $x \in S$ is said to have *character* $p \in \mathbb{N}$ if there exist $E_1 < E_2 < \cdots < E_p$ such that

$$\frac{1}{f(p)} \sum_{i=1}^{p} \|E_i x\| \approx \|x\|.$$

So if $x_1, \cdots, x_n$ are n unit vectors in $\mathbf{S}$ with a common character $p \gg n$, then

$$\left\| \sum_{i=1}^{n} x_i \right\| \geq \frac{n f(p)}{f(np)} \approx n.$$

Here $a \approx b$ means that $\left| \frac{a}{b} - 1 \right| < \epsilon$ for some $0 < \epsilon < 1$ and $a \gg b > 10$ means $\frac{f(f(a))}{b} > N$ for some $N \in \mathbb{N}$. Recall that two elements y, z in $\mathbf{S}$ are said to have the *same distribution* if there are two strictly increasing sequences $\{n_k\}$, $\{m_k\}$ of natural numbers and two sequences $\{a_k\}$ $\{b_k\}$ such that $|a_k| = |b_k|$ for all $k \in \mathbb{N}$, and $y = \sum_{k=1}^{\infty} a_k e_{n_k}$, $z = \sum_{k=1}^{\infty} b_k e_{m_k}$. Two subspaces V, U are said to have the *same distribution* if there is an isometry from V onto U such that for any $v \in V$ v and $T(v)$ have the same distribution.

For any $n \in \mathbb{N}$, let $u_n = \frac{f(n)}{n} \sum_{k=1}^{n} e_k$. It is easy to see that for any $E_1 < E_2 < \cdots < E_k$,

$$\sum_{i=1}^{k} \|E_i u_n\| \leq \frac{f(n)}{f(n/k)}.$$

Hence for any $n \in \mathbb{N}$, $\|u_n\| = 1$ and if u_n has character p, then $f(p) \approx f(n)$. We will show that for any $\epsilon > 0$, there is a rapidly increasing sequence $(p_i)_{i=1}^{n}$ such that for any $n \in \mathbb{N}$, there is a finite sequence $(v_j)_{j=1}^{n}$ in $\mathbf{S}$ such that

- the support of v_j's are pairwise disjoint;
- v_j and u_{p_j} have the same distribution;
- $\left\| \sum_{j=1}^{n} v_j \right\| \leq 1 + \epsilon$.

From that fact, it will be easy (see Theorem 6) to show that the answer to Question 1 is affirmative.

E. Odell and Th. Schlumprecht constructed a space which has neither ℓ_p-spreading model for any $1 \leq p < \infty$ nor c_0-spreading model [13]. For background information about spreading models see [2].

A Banach space X is called *minimal* (a notion defined by H.P. Rosenthal) if every infinite dimensional subspace of X contains a further subspace isomorphic to X and X is called *complementably minimal* (defined by A. Pełczynski) if every infinite dimensional subspace of X contains a subspace which is isomorphic to X and complemented in X. X is called *prime* [12] if every complemented infinite dimensional subspace of X is isomorphic to X. A. Pełczynski proved that ℓ_p, $1 \leq p < \infty$ and c_0 are prime, while J. Lindenstrauss showed that ℓ_∞ is prime (see e.g. [12]) and for a long time these were the only known examples. Recently, W.T. Gowers and B. Maurey [11] constructed more prime spaces for which all complemented infinite dimensional subspaces are of finite codimension. In [16], Th. Schlumprecht showed $\mathbf{S}$ is complementably minimal. As it was remarked in [16], the space $\mathbf{S}$ is either prime or fails the Schroeder-Bernstein property (see [6], [9]). Recall that a Banach space X is injective if for any Banach space Y containing X as a subspace, there is a bounded linear projection from Y onto X. It is known that ℓ_∞ is injective. It is natural to ask the following two questions.

Question 2. *For any sequence (n_k) of natural numbers, does $\mathbf{S}$ contain a complemented subspace which is isomorphic to $\left(\sum_{k=1}^{\infty} \oplus \ell_\infty^{n_k}\right)_S$?*

Question 3. *If the answer of Question 2 is yes, is $\left(\sum_{k=1}^{\infty} \oplus \ell_\infty^{n_k}\right)_S$ isomorphic to $\mathbf{S}$?*

A positive answer to Question 2 easily follows from our construction of ℓ_∞^n and a theorem of Th. Schlumprecht (see Lemma 2 below). A negative answer to Question 3 would have shown that $\mathbf{S}$ is not prime. Unfortunately, in this article, we will show that its answer is positive. Hence, the question whether $\mathbf{S}$ is prime or not is still open. However, our result shows that c_0 is not the only subsymmetric space X which has the property that it is isomorphic to the X-sum of finite dimensional $\ell_\infty^{n_k}$ spaces for arbitrary sequence of integers (n_k).

Recall that a non-zero vector $x \in \mathbf{S}$ is said to be an *ℓ_{1+}^n-average with constant C* if $\frac{x}{\|x\|} = \sum_{i=1}^n x_i$ for some sequence $x_1 < \cdots < x_n$ of nonzero elements of $\mathbf{S}$ such that $\|x_i\| \leq Cn^{-1}$ for all $i \leq n$.

We need the following lemmas.

Lemma 1. (Lemma 4 [10]) *Let $M, N \in \mathbb{N}$ and $C \geq 1$, let $x \in \mathbf{S}$ be an ℓ_{1+}^N-vector with constant C, and let $E_1 < \cdots < E_M$ be a sequence of intervals. Then*

$$\sum_{j=1}^{M} \|E_j x\| \leq C(1 + 2M/N)\|x\|.$$

Lemma 2. (Theorem 3 [16]) *For any* $\varepsilon_k > 0$ *with* $\sum_{k=1}^{\infty} \varepsilon_k \leq 1$*, there exists a constant* $C > 1$ *such that for any normalized block basis* (y_k) *of* (e_k) *with the following properties: there is a sequence* $r_k \uparrow \infty$ *in* $\mathbb{N}$ *so that for all* $k \in \mathbb{N}$*,*

$$\sup_{\substack{r \leq r_{k-1} \\ E_1 < E_2 < \cdots < E_r}} \sum_{i=1}^{r} \|E_i(y_k)\| \leq 1 + \varepsilon_k, \tag{2}$$

$$\text{card}(\text{supp } y_k) \leq \varepsilon_k \cdot f\left(\frac{r_k}{3}\right), \tag{3}$$

we have that (y_k) *is* C*-equivalent to* (e_k)*.*

Theorem 3. For any $\frac{1}{8} > \epsilon > 0$, there is a rapidly increasing sequence $\{p_k\}$ of natural numbers such that for any n there exist n disjointly supported elements v_j, $1 \leq j \leq n$, such that for any $k \leq n$, v_k and u_{p_k} have the same distribution, and

$$\Big\| \sum_{i=k}^{n} v_j \Big\| \leq 1 + \epsilon.$$

Proof. Let $(p_j)_{j=0}^{\infty}$ and $(q_j)_{j=0}^{\infty}$ be two sequences of natural numbers which satisfy the following conditions.

1. $f(q_0) > 20/\epsilon$ and $p_0 = 1$.
2. If q_j is defined, then $p_{j+1} = p_j q_j^{q_j^2}$.
3. Suppose that p_j is defined. By Lemma 1, there is q_j such that if x is an $\ell_{1+}^{q_j}$-vector with constant $1 + \epsilon$, then

$$\sum_{i=1}^{p} \|E_i x\| \leq (1+\epsilon)(1 + 2p/q_j)\|x\| \leq (1 + 2\epsilon)\|x\|$$

   for any $p \leq p_j^{p_j^{p_j}}$ and $E_1 < E_2 < \cdots < E_p$.

Let $\{w_{m_1,m_2,\cdots,m_j} : j \leq n \text{ and } m_i \leq q_{i-1}^{q_{i-1}^2} \text{ for all } i \leq j\}$ be any sequence of $\mathbb{N}$ such that

$$w_{m_1,m_2,\cdots,m_j} < w_{r_1,r_2,\cdots,r_\ell}$$

if one of the following conditions holds.

- for some $i \leq \max\{j, \ell\}$, $m_s = r_s$ for all $s < i$ and $m_i < r_i$;
- $j < \ell$ and $m_s = r_s$ for all $s \leq j$.

Finally, let

$$v_j = \frac{f(p_j)}{p_j} \sum_{i_1=1}^{q_0^{q_0^2}} \sum_{i_2=1}^{q_1^{q_1^2}} \cdots \sum_{i_j=1}^{q_{j-1}^{q_{j-1}^2}} e_{w_{i_1,i_2,\cdots,i_j}}.$$

We claim that for any $k \leq n$ and any interval E either $E(v_k) = 0$ or

$$\Big\| E\Big(\sum_{j=k}^{n} v_j\Big)\Big\| \leq (1 + \epsilon/4 + \epsilon/2^{k+2})\|E(v_k)\|. \tag{4}$$

Clearly, (4) implies our theorem and it holds if $k = n$. Assume that (4) holds for all $k \geq m > 1$. We will show that (4) also holds for $k = m-1$. We note that:

1. there is a vector y_m such that
   (a) $\|y_m\| = \frac{f(p_{m-1} q_{m-1}^{q_{m-1}^2})}{p_{m-1} f(q_{m-1}^{q_{m-1}^2})} \leq \frac{1+\epsilon/4}{p_{m-1}}$;
   (b) v_m is the sum of p_{m-1} successive vectors which have the same distribution as y_m.
2. $E(v_{m-1}) = \frac{f(p_{m-1})}{p_{m-1}} \sum_{i=1}^{j} e_{\ell_i}$ for some j and $\ell_1 < \ell_2 < \cdots < \ell_j$.

We shall use induction (on j above) to prove (4). First, we summarize some facts that we shall use in our proof.

Fact 1. If $E(v_{m-1}) = 0$, then by induction hypothesis, we have

$$\Big\| E\Big(\sum_{k=m}^{n} v_k\Big)\Big\| \leq \Big(1 + \frac{\epsilon}{2}\Big)\|E(v_m)\| \leq \Big(1 + \frac{\epsilon}{2}\Big)\|y_m\| \leq \frac{\sqrt{2}}{p_{m-1}}.$$

If $\|E(v_{m-1})\| = \frac{f(p_{m-1})}{p_{m-1}}$, then

$$\Big\| E\Big(\sum_{k=m}^{n} v_k\Big)\Big\| \leq (1+\epsilon)\|E(v_m)\| \leq 2(1+\epsilon)\|y_m\| \leq \frac{4}{p_{m-1}}.$$

Fact 2. We claim that $\|E(v_{m-1})\| = \frac{f(p_{m-1})}{p_{m-1}}$ implies that

$$\Big\| E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\| = \frac{f(p_{m-1})}{p_{m-1}} = \|E(v_{m-1})\|.$$

For any $r \geq 2$, and $E_1 < E_2 < \cdots < E_r$

$$\begin{aligned} & \frac{1}{f(r)} \sum_{i=1}^{r} \Big\| E_i \circ E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\| \\ \leq\; & \frac{1}{f(r)}\Big(\frac{f(p_{m-1})+4}{p_{m-1}} + \sum_{E_i \circ E(v_{m-1})=0} \Big\| E_i \circ E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\|\Big) \\ \leq\; & \frac{1}{f(r)}\Big(\frac{f(p_{m-1})+4}{p_{m-1}} + \frac{4f(r)}{p_{m-1}}\Big) < \frac{f(p_{m-1})}{p_{m-1}}. \end{aligned}$$

Therefore,

$$\Big\|E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\| = \Big\|E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\|_\infty = \frac{f(p_{m-1})}{p_{m-1}} = \|E(v_{m-1})\|.$$

Fact 3. Suppose that $\|E(v_{m-1})\| > \frac{f(p_{m-1})}{p_{m-1}}$. It is known that for any $r \geq q_{m-1}^{q_{m-1}^2}$, u_r is an $\ell_{1^+}^{q_{m-1}}$-vector with constant $1+\epsilon/2$. This implies that $E(v_m)$ is an $\ell_{1^+}^{q_{m-1}}$-vector with constant $1+\epsilon/2$ and $E\Big(\sum_{k=m}^n v_k\Big)$ is an $\ell_{1^+}^{q_{m-1}}$-vector with constant $1+\epsilon$.

Fact 4. If $\Big\|E\Big(\sum_{k=m-1}^n v_k\Big)\Big\| = \Big\|E\Big(\sum_{k=m-1}^n v_k\Big)\Big\|_\infty$, then (4) holds. So we assume that there are $p \geq 2$ and $E_1 < E_2 < \cdots < E_p$ such that

$$\Big\|E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\| = \frac{1}{f(p)}\sum_{i=1}^{p}\Big\|E_i \circ E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\|.$$

We claim that either $E_i \circ E(v_{m-1}) \neq 0$ for all $i \leq p$ or $\|E_i \circ E(v_{m-1})\| \leq \frac{f(p_{m-1})}{p_{m-1}}$ for all $i \leq p$.

First, we note that by the proof of Fact 2, if $E_i \circ E(v_{m-1}) = E(v_{m-1}) \neq 0$, then

$$\frac{1}{f(p)}\sum_{i=1}^{p}\Big\|E_i \circ E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\| < \Big\|E_i \circ E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\|,$$

a contradiction. Hence $\|E_i \circ E(v_{m-1})\| < \|E(v_{m-1})\|$ for all $i \leq p$. Suppose that our claim were not true. Without loss of generality, we assume that $E_1 \circ E(v_{m-1}) = 0$ and $\|E_2 \circ E(v_{m-1})\| > \frac{f(p_{m-1})}{p_{m-1}}$. Let $E_{2,1} < E_{2,2} < \cdots < E_{2,p_2}$ be a successive sequence of subintervals of E_2 such that

$$\Big\|E_2 \circ E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\| = \frac{1}{f(p_2)}\sum_{j=1}^{p_2}\Big\|E_{2,j} \circ E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\|.$$

Let ℓ be the smallest number such that $E_{2,\ell} \circ E(v_{m-1}) \neq 0$ and $E_2' = \cup_{j=\ell+1}^{p_2} E_{2,j}$. The proof of Fact 2 shows that

$$\begin{aligned}
&\frac{1}{f(p)}\sum_{i=1}^{p}\Big\|E_i \circ E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\| \\
<\ &\frac{1}{f(p)}\Big(\Big\|E_{2,\ell} \circ E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\| + \Big\|E_2' \circ E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\| \\
&+\sum_{i=3}^{p}\Big\|E_i \circ E\Big(\sum_{k=m-1}^{n} v_k\Big)\Big\|\Big),
\end{aligned}$$

a contradiction.

Fact 5. For any $k \leq p_{m-1}$, $\frac{f(p_{m-1} q_{m-1}^{q_{m-1}^2})}{f(k q_{m-1}^{q_{m-1}^2})} \leq \frac{f(p_{m-1})}{f(k)}$. Therefore, if $E(v_{m-1}) \neq 0$, then

$$\|E(v_m)\| \leq \|E(v_{m-1})\| + \|y_m\| \leq \|E(v_{m-1})\| + \frac{1+\epsilon/4}{p_{m-1}}.$$

Fact 2 shows that (4) holds when $j = 1$. So we may assume that $j > 1$. The proof consists of four cases.

Case 1. $1 < p \leq j$. By Fact 4, for all $i \leq p$,

$$E_i \circ E(v_{m-1}) \neq 0 \qquad \text{and} \qquad \|E_i \circ E(v_{m-1})\| < \frac{j f(p_{m-1})}{p_{m-1} f(j)}.$$

Using the induction hypothesis, we have

$$\begin{aligned} & \frac{1}{f(p)} \sum_{i=1}^{p} \Big\| E_i \circ E\Big(\sum_{k=m-1}^{n} v_k \Big) \Big\| \\ \leq \;& \frac{1+\epsilon/4+\epsilon/2^{m+1}}{f(p)} \sum_{i=1}^{p} \|E_i \circ E(v_{m-1})\| \\ \leq \;& (1+\epsilon/4+\epsilon/2^{m+1}) \|E(v_{m-1})\|. \end{aligned}$$

Case 2. $j \leq p \leq q_0$. By Fact 4, for any $i \leq p$ either $E_i \circ E(v_{m-1}) = 0$ or $E_i \circ E(v_{m-1}) = \frac{f(p_{m-1})}{p_{m-1}} e_\ell$. Note that $2p \leq 2q_0 \leq \frac{f(p_1)}{q_0} \leq \frac{\epsilon f(p_1)}{10}$. Hence,

$$\begin{aligned} & \sum_{i=1}^{p} \frac{1}{f(p)} \Big\| E_i \circ E\Big(\sum_{k=m-1}^{n} v_k \Big) \Big\| \\ \leq \;& \|E(v_{m-1})\| + \sum_{E_i \circ E(v_{m-1})=0} \Big\| E_i \circ E\Big(\sum_{k=m}^{n} v_k \Big) \Big\| \\ \leq \;& \|E(v_{m-1})\| + \frac{2p}{p_{m-1}} \\ \leq \;& (1+\epsilon/4) \|E(v_{m-1})\|. \end{aligned}$$

Case 3. $q_0 \leq p \leq p_{m-1}^{p_{m-1}^{p_{m-1}}}$. By Fact 3, $E(\sum_{k=m}^{n} v_k)$ is an $\ell_{1^+}^{q_{m-1}}$-vector with constant $1+\epsilon$. Therefore,

$$
\begin{aligned}
&\sum_{i=1}^{p} \frac{1}{f(p)} \Big\| E_i \circ E\Big(\sum_{k=m-1}^{n} v_k \Big) \Big\| \\
\leq\ & \sum_{i=1}^{p} \frac{1}{f(p)} \Big\| E_i \circ E(v_{m-1}) \Big\| + \sum_{i=1}^{p} \frac{1}{f(p)} \Big\| E_i \circ E\Big(\sum_{k=m}^{n} v_k \Big) \Big\| \\
\leq\ & \|E(v_{m-1})\| + (1+\epsilon) \sum_{i=1}^{p} \frac{1}{f(p)} \Big\| E_i \circ E(v_m) \Big\| \\
\leq\ & \|E(v_{m-1})\| + \frac{(1+2\epsilon)(1+\epsilon)\|E(v_m)\|}{f(p)} \\
\leq\ & \Big(1 + \frac{2}{f(p)}\Big) \|E(v_{m-1})\| \leq (1+\epsilon/4)\|E(v_{m-1})\|.
\end{aligned}
$$

Case 4. $p_{m-1}^{p_{m-1}^{p_{m-1}}} < p$. In this case, we have $\|E(v_{m-1})\|_1 / f(p) \leq \frac{\epsilon \|v_{m-1}\|_\infty}{4^{m+2}}$.

$$
\begin{aligned}
&\sum_{i=1}^{p} \frac{1}{f(p)} \Big\| E_i \circ E\Big(\sum_{k=m-1}^{n} v_k \Big) \Big\| \\
\leq\ & \sum_{i=1}^{p} \frac{1}{f(p)} \Big\| E_i \circ E\Big(v_{m-1} \Big) \Big\| + \sum_{i=1}^{p} \frac{1}{f(p)} \Big\| E_i \circ E\Big(\sum_{k=m}^{n} v_k \Big) \Big\| \\
\leq\ & \frac{\epsilon \|v_{m-1}\|_\infty}{4^{m+2}} + \Big(1 + \frac{\epsilon}{4} + \frac{\epsilon}{2^{m+2}}\Big) \sum_{i=1}^{p} \frac{1}{f(p)} \Big\| E_i \circ E(v_m) \Big\| \\
\leq\ & \frac{\epsilon \|v_{m-1}\|_\infty}{4^{m+2}} + \Big(1 + \frac{\epsilon}{4} + \frac{\epsilon}{2^{m+2}}\Big) \|E(v_m)\| \\
\leq\ & \Big(1 + \frac{\epsilon}{4} + \frac{\epsilon}{2^{m+1}}\Big) \|E(v_{m-1})\|.
\end{aligned}
$$

The proof is complete. □

Remark 4. *The above Theorem was first proved by the second author and independently by the first author with techniques which were later used in [1] for some asymptotic ℓ_1 spaces. Recently A. Manoussakis has the same result for more general class of spaces, containing S.*

Remark 5. *Let v_j's be the vectors in the proof of Theorem 3. We have proved that for any $m \leq n$ and $x \in \operatorname{span}\{v_m, \cdots, v_n\}$, x is an $\ell_{1^+}^{q_{m-1}}$-vector with constant $1+\epsilon$.*

By Lemma 1, for any $N, r, m \in \mathbb{N}$ and $\epsilon > 0$, there exists an m dimensional space V such that

- *$N < V$ and V is $1+\epsilon$ equivalent to ℓ_∞^m;*
- *For any $x \in V$ and any $E_1 < E_2 < \cdots < E_r$, we have*

$$\sum_{i=1}^{r} \|E_i x\| \leq (1+\epsilon)\|x\|.$$

Theorem 6. **S** *has an* ℓ_1*-spreading model.*

Proof. By Theorem 3, there exists a successive sequence $\{y_k\}$ in the unit ball of **S** such that for any k, there exist k disjointly supported elements $\{v_{k,j} : j \leq k\}$ so that for each $j \leq k$, $v_{k,j}$, $u_{p_j}/2$ have the same distribution and $y_k = \sum_{j=1}^{k} v_{k,j}$. Hence, if $N < n_1 < n_2 < \cdots < n_j$, then

$$\Big\| \sum_{i=1}^{j} y_{n_i} \Big\| \geq \Big\| \sum_{i=1}^{j} v_{n_i,N} \Big\| \approx \frac{jf(p_N)}{2f(jp_N)}.$$

So $\{y_k : k \in \mathbb{N}\}$ generates an ℓ_1-spreading model. $\square$

In the sequel we shall use a variant of the refinement of Johnson's argument for a space with a subsymmetric basis ([5], Proposition 6.3). We shall work with finite dimensional decompositions instead of block-vectors. Fix a sequence (n_k) of natural numbers. Represent any k in the form $k = 2^i(2j-1)$. Let $m_k = n_j$, i.e. we repeat each n_j infinitely many times. In the following, X, Y and V denote the spaces

$$X = \Big(\sum_{k=1}^{\infty} \oplus \ell_\infty^{n_k - 1}\Big)_{\mathbf{S}}, \quad Y = \Big(\sum_{k=1}^{\infty} \oplus \ell_\infty^{n_k}\Big)_{\mathbf{S}}, \quad V = \Big(\sum_{k=1}^{\infty} \oplus \ell_\infty^{m_k}\Big)_{\mathbf{S}}.$$

Then

$$\begin{aligned} Y &= \Big(\sum_{k=1}^{\infty} \oplus \ell_\infty^{m_{2k-1}}\Big)_{\mathbf{S}}, \\ V &= \Big(\sum_{k=1}^{\infty} \oplus \ell_\infty^{m_{2k}}\Big)_{\mathbf{S}}. \end{aligned}$$

So we have

$$V \approx Y \oplus V \quad \text{ and } Y = \mathbf{S} \oplus X.$$

Proposition 7. *There exists a complemented subspace* U *of* **S** *which is* $2C$*-isomorphic to* Y*.*

Proof. By Theorem 3 and Remark 5, we can construct a sequence (r_k) of natural numbers and a successive sequence (V_k) of finite dimensional subspaces of **S** which satisfy the following conditions.

1. For any $k \in \mathbb{N}$, V_k is a subspace of **S** which is $(1+\varepsilon)$ isomorphic to $\ell_\infty^{n_k}$.
2. $V_1 < V_2 < \cdots$.

3. For any $k \in \mathbb{N}$, $p \leq r_{k-1}$ and normalized vector $y_k \in V_k$,

$$\sup_{\substack{r \leq r_{k-1} \\ E_1 < E_2 < \cdots < E_r}} \sum_{i=1}^{r} \|E_i(y_k)\| \leq 1 + \varepsilon_k.$$

4. for any $y_k \in V_k$,

$$\text{card}(\text{supp } y_k) \leq \varepsilon_k \cdot f\left(\frac{r_k}{3}\right).$$

Let U be the space generated by $\cup_{k=1}^{\infty} V_k$. By (3), (4) and Lemma 2, for any $y_k \in V_k$ with $\sum_{k=1}^{\infty} y_k \in \mathbf{S}$, we have

$$\Big\|\sum_{k=1}^{\infty} \|y_k\| e_k\Big\| \leq \Big\|\sum_{k=1}^{\infty} y_k\Big\| \leq C\Big\|\sum_{k=1}^{\infty} \|y_k\| e_k\Big\|.$$

Hence U is isomorphic to Y. Now, let F_k be the smallest interval of $\mathbb{N}$ which contains the support of V_k. Since V_k is 2-equivalent to $\ell_{\infty}^{n_k}$ and ℓ_{∞} is an injective space, there is a projection P_k from $F_k(\mathbf{S})$ onto V_k such that $\|P\| \leq 2$. Define $P : \mathbf{S} \to U$ by

$$P(x) = \sum_{k=1}^{\infty} P_k(F_k(x)).$$

Since

$$\Big\|\sum_{k=1}^{\infty} P_k(F_k(x))\Big\| \leq C\Big\|\sum_{k=1}^{\infty} \|P_k(F_k(x))\| e_k\Big\| \leq 2C\|x\|,$$

P is a bounded projection from $\mathbf{S}$ onto U. □

Proposition 8. *The space Y is isomorphic to $\mathbf{S}$.*

Proof. We use Pełczynski's decomposition technique (see [12] or [5]). By Porposition 7, we may assume that V and Y are complemented subspaces of $\mathbf{S}$. Let W be the space such that

$$\mathbf{S} \approx V \oplus W.$$

Then

$$Y \oplus \mathbf{S} \approx Y \oplus V \oplus W \approx V \oplus W \approx \mathbf{S}.$$

Evidently, $\mathbf{S} = \mathbf{S} \oplus \mathbf{S}$. Therefore,

$$Y \approx \mathbf{S} \oplus X \approx \mathbf{S} \oplus \mathbf{S} \oplus X \approx \mathbf{S} \oplus Y \approx \mathbf{S}.$$

The proof is complete. □

Let (x_k) be a sequence of successive vectors in $\mathbf{S}$ such that for each $n \in \mathbb{N}$, x_k has the same distribution as u_{n_k} for some $n_k \in \mathbb{N}$. Let Y_1 be the space spanned by

$\{x_k : k \in \mathbb{N}\}$. It is known that Y_1 is a complemented subspace of $\mathbf{S}$ ([12], Theorem 3.a.4) and $Y_1 \oplus \mathbf{S} \approx \mathbf{S}$ ([5], Proposition 6.3). Since $\mathbf{S}$ is complementably minimal, the proof of Proposition 8 shows that $Y_1 \approx \mathbf{S}$. It is also known that if $(x_k)_{k=1}^n$ is a sequence of successive vectors such that for any $k \leq n$, x_k has the same distribution as u_{n^n}, then $(x_k)_{k=1}^n$ is $1+\frac{3}{n}$-equivalent the unit vector basis of ℓ_1^n and for each x in the span of $\{x_k : k \leq n\}$, x is an ℓ_{1+}^n-vector with constant $1+\frac{3}{n}$. By the proof of Proposition 7, we have the following.

Proposition 9. *Schlumprecht space* $\mathbf{S}$ *is isomorphic to* $(\sum\limits_k \oplus \ell_1^{n_k})_S$ *for any sequence of integers* (n_k).

A Banach space with an unconditional basis is said to have a *unique unconditional basis* if any two normalized unconditional bases are equivalent after a permutation, see [5] and for recent results see [7]. It was remarked in [7] that the example of Gowers in [8] has a unique unconditional basis. Proposition 8 (or Proposition 9) implies that S has no unique unconditional basis. This also immediately follows if one uses that S is complementably minimal and combines Proposition 6.3 and Proposition 6.4 [5].

Clearly, we have by duality that for any sequence (n_k) of natural numbers, $\mathbf{S}^*$ is isomorphic to $(\sum_{k=1}^\infty \oplus \ell_1^{n_k})_{\mathbf{S}^*}$ and $(\sum_{k=1}^\infty \oplus \ell_\infty^{n_k})_{\mathbf{S}^*}$. P. Casazza pointed out the following immediate consequence of our observations (in fact, not using their full generality). Recall that the same property for the classical spaces ℓ_p for $p > 2$, $1 < p < 2$ and $p = 1$ was shown in [14], [3] and [4] respectively.

Corollary 10. *Schlumprecht space* $\mathbf{S}$ (*resp. its dual* $\mathbf{S}^*$) *has a subspace isomorphic to the whole space and not complemented in* $\mathbf{S}$ (*resp.* $\mathbf{S}^*$).

Remark 11. *Since* $\mathbf{S}$ *is complementably minimal, most of the above results are valid hereditarily, i.e. in every subspace of* $\mathbf{S}$.

Remark 11. *Let Z be a complemented subspace of* $\mathbf{S}$ *which has a subsymmetric basis. Since Z is isomorhic to its square, then by the decomposition technique we have that Z is isomorphic to* $\mathbf{S}$.

We do not know the answer to the following question:

Question 4. *Does* $\mathbf{S}$ *have a unique subsymmetric basis?*

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