A Separable Space With No Schauder Decomposition

by

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Abstract

We combine some known results to remark that there exists a separable Banach space which fails to have a Schauder decomposition. It can be chosen as a subspace of Gowers-Maurey space without any unconditional basic sequence.

In [9] was raised the following problem (Problem 15.1, p. 494): does every separable Banach space have a Schauder decomposition? This question goes back to J. R. Retherford [7].

Recall that a sequence $\{X_n\}_{n=1}^{\infty}$ of closed subspaces of a Banach space X is said to be a Schauder decomposition of X if every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} x_n$, with $x_n \in X_n$ for every n.

Let GM be Gowers-Maurey space which does not contain any unconditional basic sequence [3]. As it was observed by W. B. Johnson, GM has in fact a stronger property, namely it is hereditarily indecomposable ($H.I.$); i.e., no infinite-dimensional closed subspace can be written as a direct sum $Y \oplus Z$, where Y and Z are infinite-dimensional closed subspaces. It is known that every block subspace of GM contains uniform copies of ℓ_1^n. This follows from the lower f-estimate and Krivine's theorem as in [8]. Then, by Szankowski's refinement of Enflo's criterion (see [6, p. 111, Remark 1]), we immediately obtain the following.

Proposition. *There exists a subspace X of GM which does not have the compact approximation property (C.A.P).*

Remark 1. For the same purpose we can as well use other $H.I.$ spaces constructed after the breakthrough of W. T. Gowers and B. Maurey. For example, there are subspaces without the $C.A.P.$ of the super-reflexive $H.I.$ spaces in [4] in the case when they contain uniform copies of ℓ_p^n for $p \neq 2$. One can also use the asymptotic ℓ_1 hereditarily indecomposable spaces constructed in [1] and [1]. The existence of uniform copies of ℓ_1^n in these spaces follows directly from the definition and one does not need to apply Krivine's theorem. Therefore, they also have subspaces without the $C.A.P.$

Corollary. *The space X is an example of a separable Banach space with no Schauder decomposition.*

Proof. Assume the contrary, i.e. X has a Schauder decomposition $\{X_n\}_{n=1}^{\infty}$.

Case 1. $\{X_n\}_{n=1}^{\infty}$ is a finite-dimensional decomposition. This is impossible since the existence of an $F.D.D.$ implies $B.A.P.$ which in turn implies $C.A.P.$ (see [5]) and this contradicts the above Proposition.

Case 2. There exists m such that X_m is infinite-dimensional. Denote $Y = [X_n : n \neq m]$. Then $X = X_m \oplus Y$ which is also impossible because X_m and Y are closed infinite dimensional-subspaces of X, X is a closed subspace of GM, and GM is $H.I.$ □

Remark 2. Clearly, the result is true hereditarily in all the above mentioned $H.I.$ spaces, e.g. we have that every subspace of GM has a further subspace which has no Schauder decomposition.

References

[1] S. Argyros and I. Deliyanni, *Examples of asymptotic ℓ_1 spaces*, Trans. Amer. Math. Soc. **349** (1997), 973–995.

[2] S. Argyros, I. Deliyanni, D. Kutzarova and A. Manousakis, *Modified mixed Tsirelson spaces*, J. Funct. Anal., to appear.

[3] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. **6** (1993), 851–874.

[4] V. Ferenczi, *A uniformly convex hereditarily indecomposable Banach space*, Israel J. Math. **102** (1997), 199–225.

[5] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, Sequence Spaces, Springer-Verlag, 1977.

[6] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II*, Function Spaces, Springer-Verlag, 1979.

[7] J. R. Retherford, *Some remarks on Schauder bases II*, Rev. Rouman. Math. Pure et Appl. **13** (1968), 521–527.

[8] Th. Schlumprecht, *An arbitrarily distortable Banach space*, Israel J. Math. **76** (1991), 81–95.

[9] I. Singer, *Bases in Banach spaces II*, Springer-Verlag, 1981.