HOMEWORK 1 SUPPLEMENT — SOLUTIONS

(1) Suppose $\frac{1}{y} = x + C$. True or $\boxed{\text{false}}$: is
$$y = x^{-1} + C^{-1}?$$
For example, with $x = C = 1$ we have $\frac{1}{y} = x + C = 1 + 1 = 2$, and so in fact $y = \frac{1}{2}$. But $x^{-1} + C^{-1} = 2$.

A correct formula is $y = (x + C)^{-1} = \frac{1}{x+C}$.

(2) Notation: in mathematics "log" means the same thing as "ln", in other words the natural logarithm or logarithm to base e.

Suppose $e^y = x + 1$. True or $\boxed{\text{false}}$: is $y = \log x + \log 1$? For example, with $x = 1$ we have $e^y = x + 1 = 2$ and so $y = \log 2$. But $\log x + \log 1 = \log 1 + \log 1 = 0$.

A correct formula is $y = \log(x + 1)$.

(3) Suppose $\arcsin y = x + \pi/6$. True or $\boxed{\text{false}}$: is $y = \sin(x) + \sin(\pi/6)$? For example, with $x = \pi/6$ we have $\arcsin y = x + \pi/6 = \pi/3$ and so $y = \sin(\pi/3) = \sqrt{3}/2$. But $\sin(x) + \sin(\pi/6) = \sin(\pi/6) + \sin(\pi/6) = 1$.

A correct formula is $y = \sin(x + \pi/6)$.

(4) A quadratic equation $\boxed{ar^2 + br + c = 0}$ can be solved by factoring the equation as
$$a(r - r_1)(r - r_2) = 0,$$
in which case the roots are $r = r_1$ and $r = r_2$, or else by invoking the *quadratic formula*
$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
Advice. Since it is often difficult to see how to factor a quadratic equation, your best bet is usually to invoke the quadratic formula.

Here are the questions: find the roots of

(i) $2x^2 - 4x - 7 = 0$:
$$r_1, r_2 = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot (-7)}}{2 \cdot 2} = 1 \pm \frac{3}{\sqrt{2}}$$

(ii) $2x^2 - 4x + 7 = 0$:
$$r_1, r_2 = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot 7}}{2 \cdot 2} = 1 \pm i\frac{\sqrt{10}}{2}$$
where $i = \sqrt{-1}$. Notice we have simplified the final answers, by using the general facts $\sqrt{AB} = \sqrt{A}\sqrt{B}$ and $\sqrt{A/B} = \sqrt{A}/\sqrt{B}$.

(5) Consider a graph $y = f(x)$. Using that $\frac{dy}{dx} = f'(x)$ equals the slope of the graph at the point $(x, f(x))$ on the graph, explain with a picture why it is that $f'(x) \approx \frac{\Delta y}{\Delta x}$, where $\Delta y = f(x+h) - f(x)$ and $\Delta x = (x+h) - x = h$, and h is small.

[Recall $\frac{\Delta y}{\Delta x}$ is called a *difference quotient*, and gives the "rise in y" divided by the "run in x". The difference quotient justifies our interpretation of the derivative $\frac{dy}{dx}$ as a rate of change of quantity y with respect to changes in quantity x.]

(6) Evaluate the following derivatives and antiderivatives (indefinite integrals):

$$\frac{d}{dx}\, x^3 = 3x^2 \qquad \int x^4\, dx = \frac{1}{5}x^5 + C$$

$$\frac{d}{dx}\, x^{-3} = -3x^{-4} \qquad \int x^{-4}\, dx = -\frac{1}{3}x^{-3} + C$$

$$\frac{d}{dx}\, \sin x = \cos x \qquad \int \sin x\, dx = -\cos x + C$$

$$\frac{d}{dx}\, \cos x = -\sin x \qquad \int \cos x\, dx = \sin x + C$$

$$\frac{d}{dx}\, e^x = e^x \qquad \int e^x\, dx = e^x + C$$

$$\frac{d}{dx}\, e^{-x} = -e^{-x} \qquad \int e^{-x}\, dx = -e^{-x} + C$$

$$\frac{d}{dx}\, e^{2x} = 2e^{2x} \qquad \int e^{2x}\, dx = \frac{1}{2}e^{2x} + C$$

$$\frac{d}{dx}\, \log x = \frac{1}{x} \qquad \int \log x\, dx = x\log x - x + C$$

$$\frac{d}{dt}\, \frac{1}{t} = \frac{d}{dt}\, t^{-1} = -\frac{1}{t^2} \qquad \int \frac{1}{t}\, dt = \log|t| + C$$

$$\frac{d}{dt}\, \frac{1}{t^2} = \frac{d}{dt}\, t^{-2} = -\frac{2}{t^3} \qquad \int \frac{1}{t^2}\, dt = -\frac{1}{t} + C$$

The best way to check your antiderivatives (indefinite integrals) is to differentiate your answer, which with any luck will get you back to where you started.