# Project I — Direction Fields — Solutions*

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Total points: 10.

Problems #1-6 are worth 0.5 points each. The answers are:

1 C

2 A

3 E

4 D

5 F

6 B

Each answer should come with a plot of the corresponding direction field, on which a solution curve should be plotted. The caption should show your name and the problem number.

**Grading guidelines.** If you only considered special cases (such as trigonometric $f$) in your solutions, then you do not get full credit. Also, notice how we illustrate the Explanations below by referring to the most relevant equations and direction fields in questions 1–6.

**Hint:** to get unstuck on a problem, plot some special cases using Iode.

8. (1 point) Suppose $y = y(x)$ solves $\frac{dy}{dx} = f(x)$.

   If $y(x)$ is periodic then $f(x)$ is periodic. (True/False, and Explain)

   Answer: True.

   Explanation:

$$\begin{aligned} y(x) \text{ periodic} &\Longrightarrow y(x) = y(x+P) \text{ for all } x \\ &\Longrightarrow y'(x) = y'(x+P) \text{ for all } x, \text{ by the chain rule} \\ &\Longrightarrow f(x) = f(x+P) \text{ for all } x \\ &\Longrightarrow f(x) \text{ is periodic} \end{aligned}$$

   Notice that this explanation applies to all periodic functions, not just to a few examples.

9. (1 point) Suppose $y = y(x)$ solves $\frac{dy}{dx} = f(x)$.

   If $f(x)$ is periodic then $y(x)$ is periodic. (True/False, and Explain)

   Answer: False.

   Explanation: For example, use Iode to examine the direction field of $\frac{dy}{dx} = 1 + \sin x$. Here $f(x) = 1 + \sin x$ is certainly periodic, with period $P = 2\pi$. But solving the equation gives $y = x - \cos x + C$, which is *not* periodic. In fact $y(x)$ gets bigger and bigger as $x$ increases (imagine the graph!).

   Incidentally, $y(x+P) - y(x) = \int_x^{x+P} y'(s)\,ds = \int_x^{x+P} f(s)\,ds$, and so we see the solution $y(x)$ will be periodic precisely when the integral of $f$ over a period-length equals zero. [Here we use that if $f$ has period $P$ then the value of $\int_x^{x+P} f(s)\,ds$ is the same for all $x$, in other words, the integral of $f$ over a period-length is the same no matter where the period begins. Exercise.]

10. (2 points) The plot of the direction field for $\frac{dy}{dx} = f(x, y)$ shows a vertically repeating pattern (e.g. Fig. F) if ...

Answer: $\ldots f(x,y)$ is periodic in the $y$-variable, in other words if for some number $P > 0$ we have $f(x,y) = f(x, y+P)$ for all $x$ and $y$.

Explanation: Suppose $f(x,y)$ is periodic in the $y$-variable, say with period $P$. Then for each point $(x,y)$ we have $f(x,y) = f(x,y+P)$, and so the slope of the line segment at $(x,y)$ is the same as the slope of the line segment at $(x, y+P)$. Since adding $P$ to the $y$-variable means translating by $P$ units vertically, we see that the pattern in the direction field must repeat itself every $P$ units vertically.

For example, in Fig. F one can see a pattern that repeats itself about every 1 unit vertically. Sure enough, Fig. F is the direction field of the function $f(x,y) = x^2 + \exp(x^2)\cos^2(3y)$, which is periodic in the $y$-variable with period $\pi/3 \simeq 1$.

**Notes.** The important point is "periodicity in the $y$-variable". Just saying "periodicity" is not enough, and just saying "trigonometric" is not enough (since there are certainly periodic functions that are not trigonometric).

**General point on explaining mathematics.** Many problems are, at heart, asking you to explain the connection between a formula and a graph. e.g. In the above problem, the task is to explain how the periodicity in the formula for $f$ leads to the repeating pattern in the graph. Keep this general point in mind, as you work on future Projects.

11. (1 point) Suppose $y = y(x)$ solves $\frac{dy}{dx} = f(x)$.

    If $y(x) \to \infty$ as $x \to \infty$ then $f(x) \to \infty$ as $x \to \infty$. (True/False, and Explain)

    Answer: False.

    Explanation: Consider the equation $\frac{dy}{dx} = 1$, which has solution $y(x) = x$, for example. Here $y(x) \to \infty$ as $x \to \infty$, but $f(x) = 1 \not\to \infty$ as $x \to \infty$.

    Roughly, this means that a solution can blow up even though its slope does not.

12. (0.5 points) Consider $\frac{dy}{dx} = f(y)$, and suppose $f(2) = 0$. What feature do you observe in the direction field, at height 2?

    Answer: All the line segments are horizontal (slope 0), when we look at points on the line $y = 2$.

Explanation: At all points with $y = 2$ we have $f(y) = f(2) = 0$, and so the slope is zero.

13. (1.5 points) Find a function $f(y)$ with the property that: for each solution $y(x)$ of $\frac{dy}{dx} = f(y)$, the limiting value $\lim_{x\to\infty} y(x)$ equals 3 if $y(0) > 0$. [Your explanation can consist of a direction field with illustrative solution curves plotted on it.]

    Answer: $f(y) = y(3 - y)$. One can plot the direction field with Iode.

    Explanation:

    Your explanation can consist of a direction field with illustrative solution curves plotted on it, showing that if the initial data $y(0)$ is positive, then the height of the solution approaches 3, as $x$ increases, while if the initial data $y(0)$ is negative then the height of the solution appears to approach $-\infty$ as $x$ increases.

    Or in words, one can observe that all the line segments above height $y = 3$ have negative slope, while those between heights $y = 0$ and $y = 3$ have positive slope, and those below height $y = 0$ have negative slope.