

Project II — Numerical Methods — Solutions*

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1. (a) (2 points) Call (t_i, x_i) the “current point”. Then $k_1 = f(t_i, x_i)$ is the slope of the direction field at the current point. Now, the point $(t_i + h, x_i + hk_1)$ would be the “next point” if we were using the Euler update formula, and so $k_2 = f(t_i + h, x_i + hk_1)$ is the slope of the direction field at this next Euler point.

Once we have found these two slopes, the Improved Euler method consists of averaging them to get $(k_1 + k_2)/2$, then stepping forward with this average slope from the current point (t_i, x_i) , arriving at the Improved Euler “next point” of $(t_i, x_i + h(k_1 + k_2)/2)$.

- (b) (3 points)

```
function xc=impeuler(fs,x0,tc);
    x=x0;
    xc=[x0];
    for i=1:(length(tc)-1)
        h=tc(i+1)-tc(i);
        k1=feval(fs,tc(i),x);
        k2=feval(fs,tc(i)+h,x+h*k1);
        x=x+h*(k1+k2)/2;
        xc=[xc,x];
    end;
```

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2. (0 points) Nothing to grade.
3. (a) (2 points) Exact solution $x = -e^t$, found by separating the variables, or by the first order linear method, or by guessing.
- (i) See Figure 1, which has $h = 0.3$.
- Features to watch out for: all curves should go through the initial condition. And students should keep the step size fixed for all the methods. They should choose a step size for which the Euler graph and the exact solution are visibly different. (If the step size is too small, all solutions might look the same.) But **take off a point** if their step size is too big to be sensible, such as $h = 1$, because they were instructed to make the Runge-Kutta graph very close to the exact solution.
- (ii) Runge-Kutta is closest to the exact solution, then Improved Euler, then Euler, although it might be hard to tell the difference between the exact solution, Runge-Kutta, and improved Euler on this problem. The Euler solution undershoots the exact solution significantly.
- (b) (2 points) Exact solution $x = te^{-\cos t} + e^{-\cos t}$, found by the first order linear method.
- (i) See Figure 2, which has $h = 0.2$, and see the remarks above.
- (ii) Runge-Kutta is closest to the exact solution, then Improved Euler, then Euler.
- (c) (2 points) Exact solution $x = -\log(\frac{1}{5} \cos 5t - \frac{1}{5} + e^{-0.8})$, found by separating the variables.
- (i) See Figure 3, which has $h = 0.1$, and see the remarks above.
- (ii) Runge-Kutta is closest to the exact solution, then Improved Euler, then Euler.

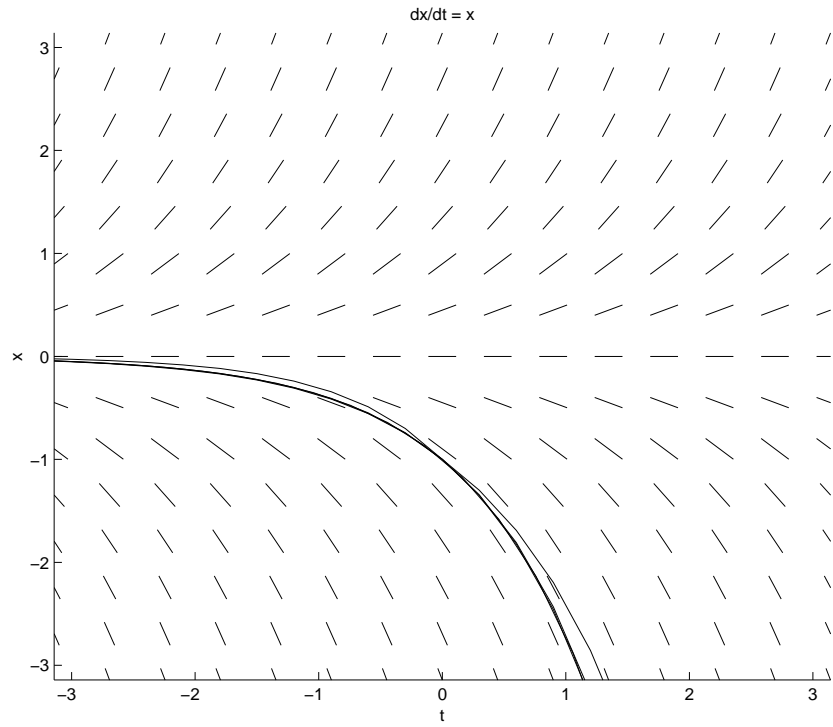


Figure 1: The plot for Problem 3a.

Student Version of MATLAB

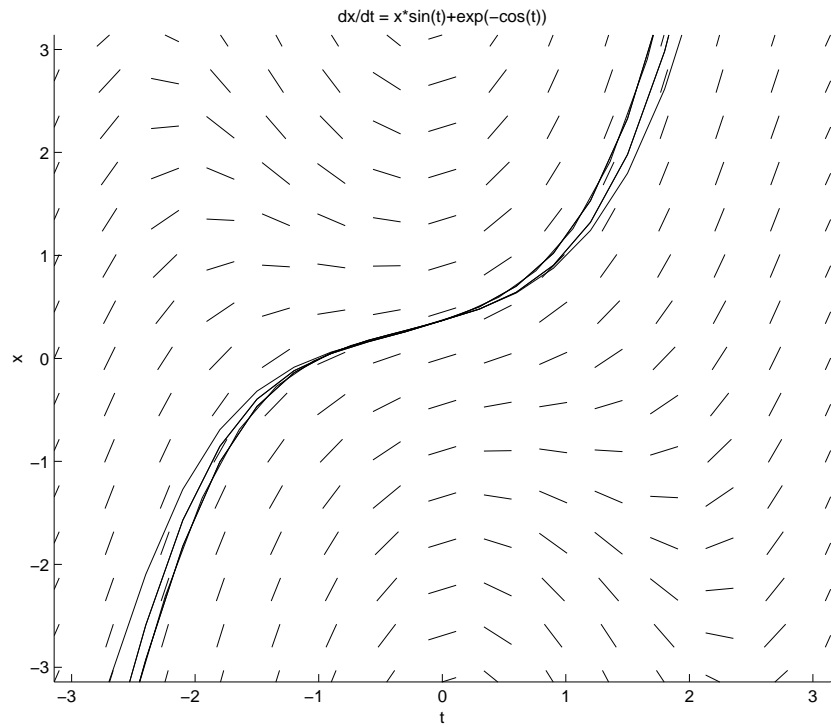


Figure 2: The plot for Problem 3b.

Student Version of MATLAB

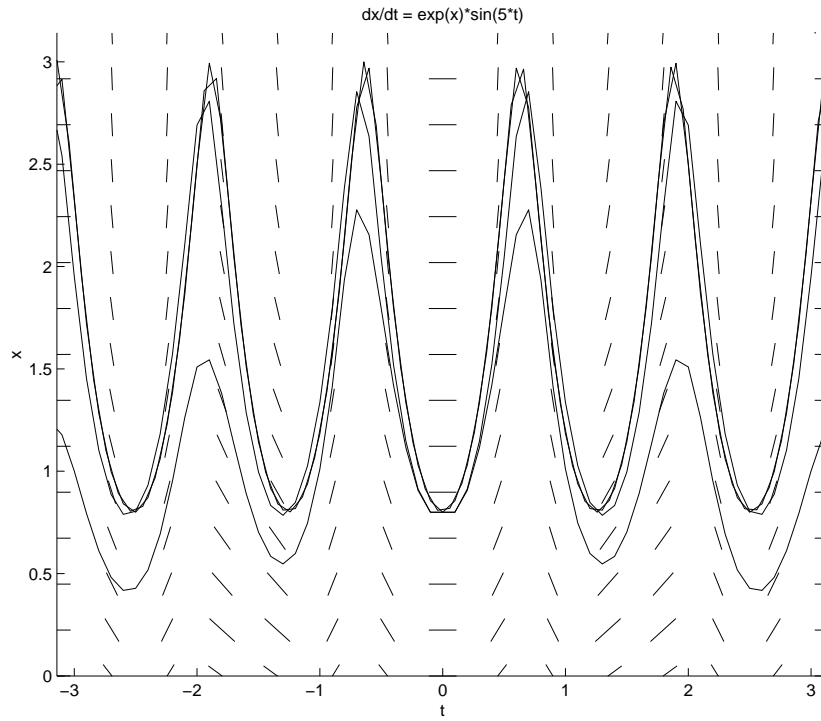


Figure 3: The plot for Problem 3c.

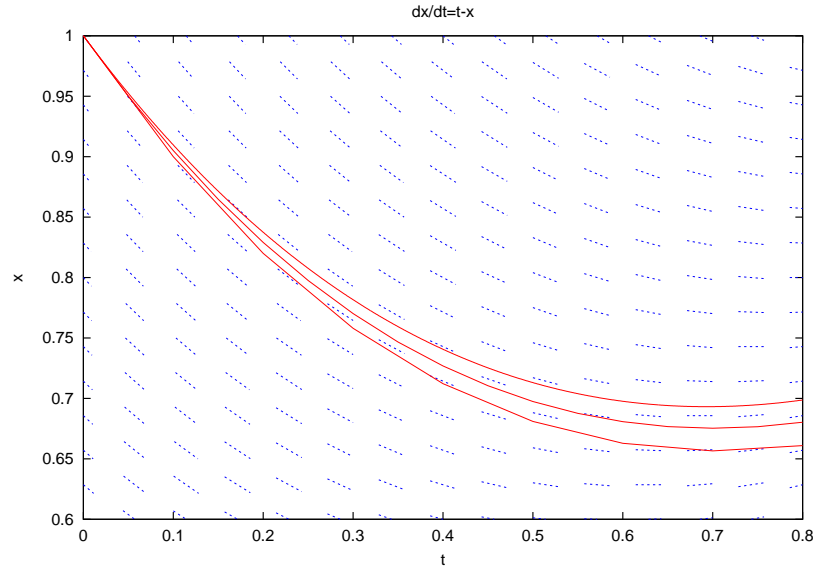


Figure 4: The plot for Problem 4(b).

4. (4 points) Consider the equation

$$\frac{dx}{dt} = t - x, \quad x(0) = 1.$$

(a) See Figure 4. The top curve is the exact solution, the middle one is Euler with $h = 0.05$ and the bottom one is Euler with $h = 0.1$.

Graphically it is clear that the ratio of the error for $h = 0.05$ over the error for $h = 0.1$ is approximately $\frac{1}{2}$. (Actually, this appears to be true not just at $t = 0.8$, but for every t -value.) So a reasonable guess is that halving the step size will halve the error, in Euler's method.

(b) See Figure 5. The bottom curve is the exact solution, the middle one is Improved Euler with $h = 0.2$ and the top one is Improved Euler with $h = 0.4$.

Graphically it seems that the ratio of the error for $h = 0.2$ over the error for $h = 0.4$ is approximately $\frac{1}{4}$. So a reasonable guess is that halving the step size will quarter the error, in the Improved Euler method.

5. (a) (1 point) See Figure 6.

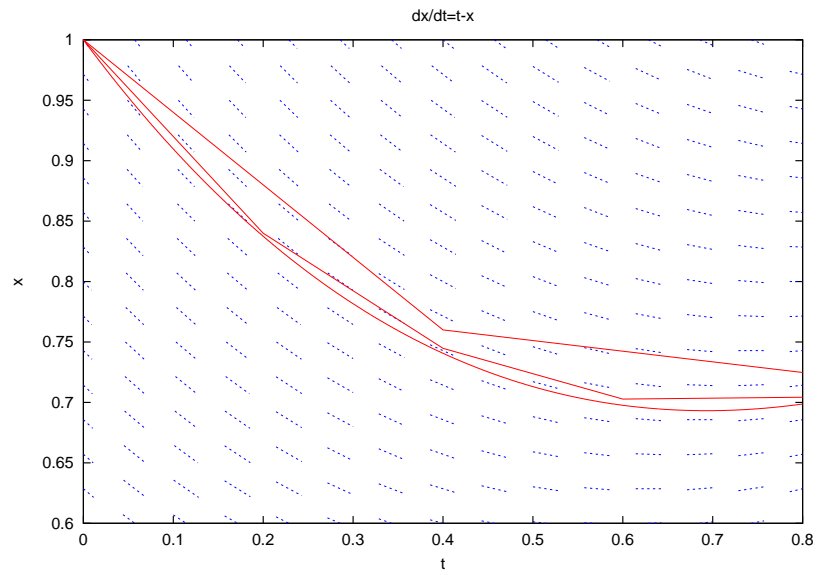


Figure 5: The plot for Problem 4(c).

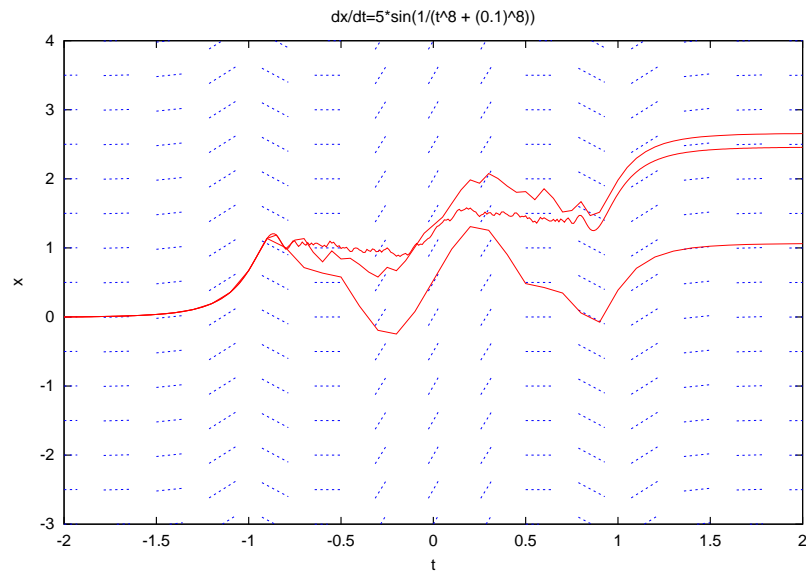


Figure 6: The plot for Problem 5(a).

- (b) (i) (0.5 points) The three plots agree for $t < -1$, then they show very different behavior for t between -1 and 1 , and for $t > 1$, the plots look roughly like vertical translates of each other.
- (ii) (0.5 points) Evaluating the term $1/(t^8 + 0.1^8)$ at $t = 0.0, 0.1, 0.2$ and 1 yields the values $10^8, 5 \times 10^7, 3.89 \times 10^5$ and 1 respectively, which shows that this function decays very rapidly from its huge maximum at $t = 0$.
- (iii) (1 point) From part (ii), we know that the value of $1/(t^8 + 0.1^8)$ quickly falls from 10^8 to approximately 1 , as t increases from 0 to 1 . The period of the sine function is 2π , and so the composition of \sin with $1/(t^8 + 0.1^8)$ goes through approximately $10^8/(2\pi) = 1.59 \times 10^7$ oscillations as t ranges from 0 to 1 . In other words, $\sin(1/(t^8 + 0.1^8))$ **oscillates very quickly**. The Iode plot of $\sin(1/(t^8 + 0.1^8))$ confirms this.
- (iv) (2 points, and grade hard) In part (iii), we saw that the slope function $\sin(1/(t^8 + 0.1^8))$ goes through approximately 1.59×10^7 oscillations on the interval between $t = 0$ and $t = 1$. Our numerical methods, when applied with step size $h = 0.01$, only look at about $1/0.01 = 100$ points in the interval between 0 and 1 , so that they miss most of the rapid oscillations that are supposed to be occurring in the slope of the exact solution curve. Hence, none of the three numerical solutions computed in part (a) is likely to be close to the exact solution. This also explains why the numerical solutions do not seem to follow most of the line segments in the direction field, when $-1 < t < 1$. Zooming in won't make much difference to these difficulties, until you have zoomed in a *very* long way.