

Project III — Solutions*

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A complete solution for this project consists of: **written answers** for problems 1ac, 2ac, 3ac, 4ab, 5a–f, and **Iode plots** for problems 1b, 2b, 3b.

Problems

The display parameters should be $0 < t < 6$, $-5 < x < 5$, for all of the following numerical experiments.

1.

$$(1)x'' + (1.5)x' + (.5)x = 0.$$

(a) *Solution.* $x = c_1e^{r_1t} + c_2e^{r_2t}$ where $r_1, r_2 = -1, \frac{1}{2}$.

(b) Students should attach a clearly labelled graph. Every solution curve should decay.

(c) 100% decay. The general solution explains this because $r_1 < r_2 < 0$, and so $e^{r_1x} \rightarrow 0$ and $e^{r_2t} \rightarrow 0$ as $t \rightarrow \infty$, giving $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

2.

$$(1)x'' - (\sqrt{3})x' - (.25)x = 0.$$

(a) *Solution.* $x = c_1e^{r_1t} + c_2e^{r_2t}$ where $r_1, r_2 = (\sqrt{3} \pm 2)/2$

(b) See above. Every solution curve should grow.

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(c) 100% growth. Write $r_1 = (\sqrt{3} - 2)/2$ and $r_2 = (\sqrt{3} + 2)/2$, so that $r_1 < 0 < r_2$. Then the general solution gives

$$x = (c_1 e^{(r_1 - r_2)t} + c_2) e^{r_2 t} \rightarrow c_2 \cdot \infty \quad \text{as } t \rightarrow \infty,$$

and so the limit is $-\infty$ if $c_2 < 0$ and is $+\infty$ if $c_2 > 0$. Note the remaining case $c_2 = 0$ is certainly possible, but does not occur in practice because the initial conditions are chosen randomly and it would take a miracle for c_2 to exactly equal zero.

3.

$$(1)x'' - (.5)x' + (.5)x = 0.$$

(a) *Solution.* $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ where $r_1, r_2 = (1 \pm i\sqrt{7})/4$. A better answer is $x = e^{t/4}(c_1 \cos(\sqrt{7}t/4) + c_2 \sin(\sqrt{7}t/4))$.

(b) See above. Every solution curve should grow and oscillate.

(c) 100% growth while oscillating.

4. a With pencil and paper, find the general solution of the equation

$$x^{(4)} + 16x'' + 100x = 0$$

(*Hint.* $r^4 + 16r^2 + 100 = (r^2 + 2r + 10)(r^2 - 2r + 10)$).

Solution. The characteristic equation is

$$0 = r^4 + 16r^2 + 100 = (r^2 + 2r + 10)(r^2 - 2r + 10) = 0,$$

by the Hint. The roots of the two quadratic factors are

$$r_1, r_2 = -1 \pm 3i, \quad r_3, r_4 = 1 \pm 3i,$$

by the quadratic formula. Hence the general solution is

$$x(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t) + e^t(c_3 \cos 3t + c_4 \sin 3t).$$

b Predict what percentage of solutions with randomly chosen initial data will grow while oscillating, and what percentage will decay while oscillating. Explain.

Solution. 100% of solutions with randomly chosen data should grow while oscillating. The reason is that the decaying terms

(which have e^{-t} in them) will approach zero, while the growing terms (which have e^t in them) will continue to grow and hence will dominate. To make this precise, we can pull out e^t as a common factor:

$$x(t) = e^t [e^{-2t}(c_1 \cos 3t + c_2 \sin 3t) + (c_3 \cos 3t + c_4 \sin 3t)].$$

This expresses $x(t)$ as a growing amplitude e^t times a term that is approximately equal to the oscillatory term $(c_3 \cos 3t + c_4 \sin 3t)$, when t is large.

5. *Conclusions.* Consider a second order linear constant coefficient homogeneous equation having roots r_1 and r_2 for its characteristic equation. Summarize your conclusions about the behavior of the solution $x(t)$, for randomly chosen initial data, when:
- a $r_1 < r_2 < 0$: decays
 - b $r_1 < 0 < r_2$: grows
 - c $0 < r_1 < r_2$: grows
 - d $r_1, r_2 = \alpha \pm \beta i$ and $\alpha < 0$: decays (while oscillating)
 - e $r_1, r_2 = \alpha \pm \beta i$ and $\alpha = 0$: oscillates
 - f $r_1, r_2 = \alpha \pm \beta i$ and $\alpha > 0$: grows while oscillating