Project IV Solutions — Fourier Series*

1 Solutions

#1 Fill in the table:

$f = 2\pi$-periodic extension of:	jumps of f, for $-\pi < x \leq \pi$	corners of f, for $-\pi < x \leq \pi$	Top harmonic N value for naked eye convergence ?	Relative error when top harmonic = 25?
$(x^2 - \pi^2)^2$	—	—	$6 - 12$	$\approx 10^{-4}$
$\|x + 1\|$	π	-1	—	$\approx .2 - 1$
$e^{-x}\text{sign}(2 - x)$	$\pi, 2$	—	—	$\approx .2 - 1$
x^2	—	π	$15 - 25$	$\approx .01 - .05$
$\sin((2/\pi)x^2 + \pi/2)$	—	—	$6 - 12$	$\approx 10^{-4}$
$x^3 - \pi^3 \sin(x/2)$	—	—	$6 - 12$	$\approx 10^{-4}$
$(x - \pi) \sinh(x + \pi)$	—	π	$15 - 25$	$\approx .01 - .05$
$(\log(x + 6))^2$	π	—	—	$\approx .2 - 1$
$\cos(\frac{1}{2}(x - 1)^2)$	—	π	$15 - 25$	$\approx .01 - .05$

#1 Grading guidelines:

- Correctly identify functions 2, 3, and 8 as having jumps, functions 4, 7, and 9 as being continuous but having corners, and the rest as continuous with continuous derivatives (no jumps and no corners).
- To get full credit, the answers should be good enough to indicate that the student took the time to understand the definitions of "naked eye convergence" and "relative error", and then did the work and took some care in making the observations. The answers do not have to be exactly those in the table above.
- Students can easily miss the jump in $e^x \text{sign}(x+2)$ at $x = -2$ ... but it really is there!

#2 Grading guidelines:

a Compare and contrast the naked eye convergence you observed in

  i functions with jumps (discontinuities).
  For these, it appears that we never really get naked eye convergence. There is always some visible error around the points where the function jumps. And in any case, the convergence is certainly nowhere near as good as it is for continuous functions (see next answer).

  ii functions that are continuous but have corners (that is discontinuities in the first derivative.)
  Naked eye convergence typically occurs when $N = 15 \sim 30$. For $N = 8 \sim 10$ the partial sums and the function are already indistinguishable away from the corners in the function, but in order to get good convergence at the corners, we need more terms in the Fourier series.

  iii functions that have no jumps and no corners (continuous, and with the first derivative continuous as well).
  In this case naked eye convergence typically occurs for $N = 6 \sim 10$.

  So we see a hierarchy: functions with jumps never give naked eye convergence, functions with no jumps but with corners give naked eye convergence for reasonably large N-values, and functions with no jumps and no corners give naked eye convergence already for rather small N-values.

b Compare and contrast the relative error you observed in

  i functions with jumps.
  The relative error for $N = 25$ is typically large, roughly .2 to .5 or larger.

  ii continuous functions with corners.
  The relative error for $N = 25$ is smaller, typically on the order of 10^{-2}.

iii functions with no jumps and no corners.

The relative error for $N = 25$ is typically really small, on the order of 10^{-4}.

We see the same hierarchy as before, in the answers.

CAVEAT. Technically speaking, the above conclusions are what one would expect to observe provided the functions are "piecewise smooth", and provided the jumps, corners and slopes are not too extreme. Precise theorems about convergence are established in higher mathematics courses.

c For functions with jumps, discuss more carefully the kind of convergence you observed: can the Fourier series be said to converge at all? What happens to the error away from the discontinuities?

Solution.

The series does converge so long as we stay away from the jumps, in the following sense. Suppose f is a function that jumps only at $\pm\pi$. If we look in the interval say from -3 to 3 (where the function is continuous), we will see that the partial sums approach f (meaning the error gets small) as N gets large. If we look in a larger interval, say from -3.1 to 3.1, we will see the same thing, although we will have to take N even larger before the partial sums are a good approximation of f in this interval…

d Suppose that f is a function with both a jump and a corner. What would you predict will be the relative error for f when the top harmonic = 25? Explain your answer. (If you think it is impossible to predict, explain why.)

Solution.

Here's what we were trying to get at with this question: you have seen that functions with corners behave one way, and that functions with jumps behave a different way. What about a function with both a corner and a jump? Which way does it behave?

The answer is: we expect the relative error may be $.2 \sim .5$. Reasoning: The relative error is an indication of how hard it is to approximate the function with a Fourier series. The fact that the relative error is always large for functions with jumps indicates that it is always hard to approximate these functions by a Fourier series. This remains true whether or not there is a corner present, and so the relative error should be comparable to what we have observed in the examples of functions with jumps, regardless of whether or not there is a corner.

Of course, this is only a general rule, and if the jump is very small compared to the size of the function, then the error might be dominated by the error at the corner. But generally, a jump causes more trouble for the Fourier series than a corner does.